ALL COLORS OF PHYSICS

TAMÁS SÁNDOR BIRÓ

GINTROPY

A FICTION ON INEQUALITY

HUN REN | WIGNER

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Tamás Fülöp, Anikó Kutnyánszky, Ilona Verle

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Luca Gulyás-Szabó

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Péter József Lévai

HUN-REN Wigner Research Centre for Physics
Konkoly-Thege Miklós út 29–33, H-1121 Budapest, Hungary

About the Author

Tamás Biró received the master's degree in Physics and Biophysics in 1980 and the Ph.D. in 1982 from Eötvös Loránd University, Budapest. He worked on theoretical heavy ion physics, in particular on strange quark production in quark-gluon plasma at the KFKI Research Institute for Particle and Nuclear Physics in Budapest; then, he continued, as researcher, at the Niels Bohr Institute in Copenhagen, at the GSI Centre for Heavy Ion Research in Darmstadt and at the Institute for Theoretical Physics at the University of Giessen. Returning to Budapest in 1994, he has been Vice Director of the Institute for Particle and Nuclear Physics at the Wigner Research Centre for Physics from 2013 to 2019. He invented the color rope model for describing the early phase of ultrarelativistic heavy ion collisions in 1983, took part in studies of chaotic dynamics in strongly interacting non-abelian gauge fields (1991–2004), and then he developed a fable for statistical physics of complex systems and non-extensive thermodynamics. Now he leads a nanoplasmonic fusion experiment, NAPLIFE, at the Wigner RCP (2020-2025). He collaborated among others with colleagues at Duke University in North Carolina, USA, at Bergen University in Norway, at the Universities of Cape Town and Johannesburg, South Africa, at the Central China Normal University in Wuhan, China, at the Yukawa Institute in Kyoto, Japan, at the UBB Cluj, Romania, and at TU Wien, Austria. He is external faculty at the Complexity Science Hub, in Vienna. He edited the Acta Physica Hungarica and was Editor in Chief of EPJ A (Hadrons and Nuclei).

Other English books written by the Author

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Pre-explanation

This book contains fiction. A fiction about scientists and science, yet it is not a genuine science fiction novel. The purpose of this writing is to bring university math models of statistical physics and some underlying thermodynamics and informatics concepts close to those readers who would be repelled by the usual formalism. That is crafted through a story, a meeting and collision of various characters doing science, and discussing their ideas.

Still, unlike in the renown dialogues by Galilei, the aim is not ideological, not trying to convince about a selected philosophy against another. Certainly, the world-view of the author shall become more or less transparent for most readers. Would it be a rational goal to decrease inequality by force? Can an answer to this question be looked for via abstract and oversimplified models of the human society, comprised into a handful of equations? Can truth be bought by money?

Such questions may interest many, even those who would not like to follow mathematical derivations, who somewhat dislike too much theory in life. Nevertheless, following another ideal, those mathematical equations and derivations are excitingly interesting on their own right. No scientist would agree to miss them: They are enjoyable. Therefore a new technique is tested by this book, that of hypertext jumps to and back from the formula parts.

As a final outcome here is a textbook, with mathematics not spared out, and still offering a continuous story to read. All the formulas can also be read at the end of the story, as an appendix. There I have tried to insert a few sentence citation of the part of the story where they fit in.

It remains to hope that this story will be able to ignite the phantasy of the distinguished Reader, as well as the underlying discussion of inequality and entropy growth concepts. The story itself should simply entertain without any hidden purpose behind. Any resemblance of characters appearing in this novel to living persons is accidental.

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Contents

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Chapter 1

Seven Scientists

Collecting an international research team for a private project.

Fred

Fred was travelling on the metro. Being a mathematician he did go to his workplace not too early, so he could have a seat and avoided the mass in the rush hour. The Budapest metro lines connect the outskirts of the 2-million city with the center, following a rather radiate pattern, no cycles, no arcs. The first line, not so deep as the others, is the Millenium line, set in function in 1896. In the same year Hungary has celebrated its one-thousand-year anniversary of settlement in the Carpathian basin, a little more than hundred years before constituting a Christian State in 1000 AD. The name day of the first Christian King, Stephen I, is still a national holy day. Transmuted into a holiday, so called red letter day, in the socialistic era. The current phrase in 1970 was "Constitution Day".

M1 was the oldest line, "First on the Continent" the propaganda said, declinating the fact that it was second in Europe, after London. The newer lines, the red and blue ones, called M2 and M3, were originally built with Russian help and they reminded more to the Moscow style underground railway. In the wagons there was a plate, "Mitishinskii Zavod". Indeed, in Moscow and in some other cities, like Paris, France, or Vienna in Austria, there were circular lines, too. So, if a map of metro lines were made by chance, randomly choosing new lines and crossing points, what is the expected degree of connectedness? What if chance is not blind, and goddess Fortuna shows some preference? And such a preference is always in favour of the already lucky ones... another dysfunction of our Universe...

We know that the answer is exponential in the first case, and a power law tailed connection number ("node degree") distribution in the second case with the positively biased preference, but could we derive this result from a simple dynamics? Yes, we are looking for mathematical laws for social phenomena, by doing this... Fred was deep in such thoughts when he left the carriage at Deák tér to change line. Then suddenly he felt a strong grip on his arms. From left and right, and immediately his foots lost contact with the floor.

“We have got you, Weasel!”

“What, weasel? This must be a mistake!”

“All say that. Come with us, without fuzz.”

Fred was carried to a huge black car with darkened windows. *“This is like a movie commonplace,”* he thought. They drove fast and not halting at any crossing for a while. No word was spoken. *“Perhaps they want to give me time to feel my fear. It can work.”* Finally the car stopped in an underground garage. One of the gorillas drew a black masking sack over Fred’s head, and he was escorted to some room, by elevator and corridors, and several turns and elevator again. Probably this was not the most direct way to reach the final place.

When the mask was pulled away, Fred had to shut his eyes from the sudden brightness of the illumination. It took a few seconds to accommodate. A man, dressed in an elegant black suit and a red tie, wearing glasses with thick, black plastic frames – as it was the fashion in the 1960-s – sat behind a big and solid-looking table. Beside stood another, younger man, also in suit, his gestures by standing there radiated his readiness for services for the boss on the slightest sign of a demand. The room had no window – it well might have been underground. The smell in the air was neutral, a slight noise of a working air-conditioner could be sensed just at the hearing-threshold. The boss told something in a language unknown to Fred. The younger one, probably a *Dolmetscher*, spoke with some nervous intonation:

“Herr Doctor, we apologize for the circumstances you are brought here. But, and we hope you will soon understand, very serious problems made us to act rapidly. Our future society is at stake.”

“Ah, hmm, ughhh...”

“The future of the whole mankind is in danger. Not less is at stake than whether we will mutually destruct each other.”

“Oh, you mean the MAD[1] will not stop us having a nuclear world war?”

“No, it is more than that. But whatever you will be informed further, you have to sign keeping all secret.”

“What if not? Oh I see. Forget this childish question.”

[1]MAD= Mutually Assured Destruction, a phrase and strategy in the era of cold war.

“Very well, your fame is not without ground being a sober thinker. You have to know, that if you join our case we pursue, you will have to quit your connection to your present life. Including family, girl-friend, whatever you had so far. Our project is not only secret, it takes place in another era ... somewhere else.”

This is going to be weird.

“What d’you mean by ’another era’? Do not feed me with sci-fi time travel.”

“It is not like that. However, please do not forget, that we all travel in time... continuously. From the past to the future. And not making jumps, at least not on the human size level of our senses. The demand is that you do not return to your home and to your acquaintances if you hired.”

“Well, what the heck... You promise me I will not be alone, and I will have a possibility to work on mathematical problems?”

“Problems we can predict. The rest is up to you and the team you will join.”

“Do I get some time to make up my mind?”

“Of course!”

Three heartbeat seconds silence.

“Now the time is over.”

“Thou measure thy givings with miser hands, sith thou are a mighty lord[2]. Very well, I am in.”

The boss nodded and Fred was brought to another end of the underground facility. This time without eye-binding. After half an hour walk and a short journey on a rapid, narrow trail railway they arrived at another locked door. The guard checked with a handy device Fred’s retina and then pushed some numbers on a code tablet. The door opened to a halfway illuminated long corridor.

Fred moved on. The light followed him – the movement sensors worked in a perfect silence. Then he arrived at a bigger room, it looked like a laboratory. Beyond the mess of tablets, screens and thousands of small instruments a big device ruled the room: looked like a very costly accelerator or an experimental fusion toroid. Its door opened and he was invited to step in by a sweet female voice from inside. Exactly this was the thing he could not resist.

[2] That is a citation from the "Tragedy of Man", a 19th century Hungarian play by Imre Madách.

Rudolph

It was a beautiful autumn morning. Blue sky, no clouds, the sunrise igniting with glittering golden lights the high roofs in the City. Rudolph decided to take a walk before breakfast at the long alley along the river Main. It was a frequented path, perfect for an early sunbath in the morning - anyway a possibility which occurs here definitely less than the half of a year. He started his walk at the Paulskirche, went to the river bank, turned westward, and enjoyed the warmth and light coming from behind. The air was fresh cold, the night was freezing, the leaves on the trees guarding the alley reflected the light in high yellow and red colors. Everything was prepared to have a great day, which fact naturally raised the mood of Rudolph. However, in small, almost unnoticed steps.

His decision to visit a newly opened coffee house nearby was quickly made. The smell of fresh croissants, the vista of small tables and chairs in the front of tiny shops on the curly streets, the morning noise made by cooks and maids inside kitchens by preparing French style breakfasts all made him to feel an irresistible attraction towards to settle. He has chosen randomly a corner place, warmed by the rising sun, ordered coffee with milk, hot, and a slice of butterstollen. He took the newspaper and started to read over the titles.

Nothing important, nothing that would move the world. *Nichts weltbewegendes.* Kaiser Wilhelm I died this Spring, and in the Summer Wilhelm II followed him as German emperor on the throne. Jack the Ripper continued his murdering among prostitutes in London. Berta Benz made an autodrive from Mannheim to Pforzheim without stopping with the new invention of his husband, Carl. A moving picture show was introduced by Louis le Prince in Paris, the *"Roundhay Garden Scene"*. The Frankfurter Börse raised, everyone predicted further bullish days. Rudolph realized that he was reading a chronicle of the year 1888.

The sun began to give more warmth. How strange! How energy converts into heat raising the temperature of the skin, of the head and of the objects around: chair, table, coffee-cup. Is heat also energy? Rudolph smiled. He was among the few who knew what heat really is. It is motion energy. Just this motion is not spectacular, not to be seen by naked eyes. For that matter also not with microscopes. The motion of atoms and molecules, collisions, rotation, oscillation, – all microscopic. Heat is like work, responsible for changes in the total energy. It is just not exerted by big forces, acting from outside the bodies; it is exerted innerly.

“*More than that,*‘ continued Rudolph his thoughts, ’*more than that, as the work can be accounted for a pure change of potential energy related to conservative force fields, the heat should also be related to a change in something like a potential field: this something must have a definite value at each space point at a given time.*”

“Entropy makes the world bowl around,” he said half loud.

“Can one make a halt to that?‘ The voice came from the left, from another little table. The gentleman in a black coat was taking a sip of his coffee slowly and carefully. ’I mean, the entropy growth?”

“Excuse me. I invented the phrase ’entropy’. It means ’in the place’, a quantity associated to points in a space of thermal parameters. I am a professor in physics.”

“I beg your mercy, Herr Professor. But why should it grow?”

“I’ll explain it to you. At least in terms of a layman... I suppose you are, are not you?”

“Not exactly a layman, but I surely would enjoy an explanation in layman terms.”

“Well, this is a pessimistic version of the ’no free lunch’ principle. The first law of thermodynamics says that the total energy can be changed only by doing work or exchanging heat. The second law says that you cannot avoid making heat by any working process. Waste is compulsory, because entropy grows.”

“Oh, how it is then that cooling cupboards work? Ice cube in the whiskey?”

“The growth of entropy happens in closed systems. It is valid for the whole universe at the end. At the heat-death everything will have the same temperature and all energy changing processes will stop.”

“Wow, I hope not today.”

“Nope, the universe is huge, accordingly it has to take a long time.”

“Herr Professor, you might find it strange, but I would like to make you a business offer.”

Rudolph deliberated quickly. Some extra appanage to the university teacher’s income could always be welcome.

“I am listening.”

“My mandators, who would not like to make their names public, I hope you accept this, are interested in building a research team. We are looking for bright minds without prejudice. The team needs one or two experienced members.”

“You are pointing to my age.”

“No, not directly. Rather to your fame in your area of expertise in making mathematical models of fundamental processes. Since you could make calculations with consequences for the whole Universe, possibly you will not mind to speculate about theories of our future society.”

“You just have gained my interest. Please, go on.”

“I cannot tell you details right now. This evening there will be a secret meeting at this address.‘ He gave Rudolph a small paper card. ’We hope you will be able to come.”

“Hmm. Bahnhofsgasse? Not the best district for a gentleman... you know what I mean?”

“No offense taken. Our enterprise needs the twilight to stay secret... at the present stage. Nonetheless, if you agree to join, you will have to cease your past life, and break your personal, familial connections. In exchange I promise you challenge and adventure – perhaps beyond your present imagination.”

“An impertinently unique business offer. So Bahnhofgasse 13, at 19:00 tonight.”

When Rudolph turned the man was already gone. The sun started to hide behind a grey cumulus, and a sudden chill was felt. The coffee became unenjoyably cold. He stood up, whirled some money onto the table, and left.

Rudolph was living alone since his wife, Rosaline, died two years ago. He continued his life, his research in thermodynamics and his teachings at the university, but all this more like a routine. His principles included the sense of duty, and he sympathized with the Christian maxima: *"you have no right to end life, because you did not make it"*, so even in his most depressive hours the thought of suicide avoided him. At least this horrifying idea was buried under the firmness of his expressions he wore on his face lately.

Making up his mind for the new adventure, and remembering by this occasion to his favourite readings from Jules Verne, who depicted in his novels that art of scientists who were matchless cases for a mixture of naïvity and bravery due to ignorance of the circumstances, he packed a little rucksack with necessary drink and food reserves, warm socks and wore a cosy cloth, as if he would go for a Sunday afternoon excursion into the hills of Taunus. Exactly at 18:59 he was at the address given him previously.

The entrance of the old house looked somewhat demolished but solid. He knocked on the door. After ten seconds he heard footsteps and clirring of a bunch of keys. It was already dark, the street lamps left this place in shadow.

“Who are you?”

He told his name. Then he showed the card he received that morning. The moon appeared and the night-guard could see the symbol on the business card. He opened the door.

“Step in, Herr Professor. You are already waited for.”

Rudolph followed the lead through a small court to a staircase, up one storey, and then to the end of a long balcony-like inner corridor. They entered in a twilit room, looking like an old salon, with a sofa, some chairs, a little tablette, smoke of cigars in the room. From the sofa a wealthy looking man stared at him.

“Are you sure this is our man?” he asked someone in the dark shadow. The other must have nodded affirmative. The man stood up and signed to the servant, who brought elegant glasses filled with champagne on a silver plate.

“Welcome, Herr Professor!” the man toasted and they all drank.

Then Rudolph felt weak knees, dizziness and the world become entirely black in front of his eyes. He fainted.

Ludvick

One, Two. One, Two. Step after step. Three, Four. Ludvick enjoyed the workout by jogging. He crossed smaller roads in the villa district and in a few minutes arrived at the Türkenschanzpark, a nice, somewhat hilly little park in the North of Vienna. This park was planted with rare trees, alike a botanical garden, a pond in the middle, curvy paths all around. Also an outdoor cafe and two playgrounds for children were included. A nice design.

At some turns, decently placed in the front of lush bushes, stone portraits and busts of once-upon-a-time famous men. In this early hour, just after sunrise, very few other people were seen. One or two other runners, a pair leading a dog on a leash, some cleaners sweeping the pavement. A small civilized Eden – he sighed.

The path took an uphill part. Ludvick did not slow down his pace, but took smaller steps and speeded up his respiration. At such tracks he could feel more muscles, from the calves through the bottom up to the lower back. This was the better part of the exercise. Then at the lighter part with almost no relative elevation he took longer steps, automatically. The movement and the monotony were triggering his thoughts.

Yesterday he visited the Zentralfriedhof, the central cemetery. As a physicist he was most impressed by Boltzmann’s tomb, although he sacrificed one or another glimpses to the monuments of music composers from the 19th century Vienna, too. Boltzmann’s gravestone is famous of a formula being carved onto it:

$$S = k \log W. \tag{1.1}$$

For what stands the W for? Some say, for "Wahrscheinlichkeit", the German phrase for probability. The choice of different phrases is already characteristic. Germans say it is a "shine", a virtual image of something which looks true or real. Practically thinking Englishmen call it something, which can be probed, can be tested. *Probability can be defined only with some probability* – came the citation from Kolmogorov, a beginning of 20th century Russian mathematician to Ludvick's mind. Then suddenly enlightening happened:

"It cannot be!"

He might have murmured it half loud. Any probability is smaller than one, but the entropy, S, is positive. The logarithm of numbers smaller than one are all negative. Something is wrong with this legend. He felt amused. Indeed, thinking of the *permutation entropy*, W must stand for $N!$, for the number of ways N elements can be exchanged, re-ordered. If a physical system has N equally probable microstates, depictable by small, uniform boxes in a big abstract volume, then $N! = N \cdot (N-1) \cdot \ldots 2 \cdot 1$ possible arrangements give us the same macroscopically sensed state. *Oh, this might be a good, pedagogical introduction to the classical entropy formula in my class today!* This thought raised his mood. Today's jogging raised his level of serotonin, as it should have.

After getting home, taking a shower, changing cloths, he run to the U-Bahn. Taking off at Karlsplatz he walked into the impressive Jugendstil building of the university, where he was giving a class this day. Indeed, he presented a derivation of Boltzmann's entropy – probability formula based on the permutation.

→ permutation entropy page 118

After finishing with the lengthy formulas, Ludvick took a deep breath and turned towards the audience. He was searching for the looks on the faces, whether was it as clear as he told it to himself. He was disappointed. Nobody seemed to be excited by this. He tried to explain:

"This derivation shows, that Boltzmann's logarithm formula is only valid, if the sum of probabilities is exactly one."

"Dr. Holzmann, how can it be else?" asked a student from the middle.

"It cannot, if a model takes all possible outcomes of microstates into account. However, it is not a valid formula for a partial knowledge about the microstates."

Time was over, Ludvick left the lecture hall after all the students streamed out. He was feeling a growing amount of disappointment. He almost overrun an elegantly dressed, middle aged, blondine at the corridor. She looked so great that Ludvick felt rebucked.

"Oh, I am sorry!" he murmured and tried to avoid the collision. The woman addressed him.

"Dr Holzmann? Ludvick Holzmann?"

He nodded. He was not able to talk.

“I would like to talk to you. I hope you do not mind joining me for a chat at lunch in the faculty loungue?”

This alerted Ludvick. Might she be from the Human Resources Department? Did he say something politically incorrect lately? Confused, he finally pressed out:

“I’ll do with delight, Miss...”

“Ms Kunz, my name. But do not mind it. I am to talk to you in the name of a special foundation, called ’Our Future Society’.”

She hooked her arms into Ludvick’s and started to walk and talk.

“We would like to make a special offer for you, a well-donated research work.”

“How could I let such a chance go?” tried Ludvick to flatter her. No effect.

“Look, it is an important research with selected participants, on a place without undue disturbance. You shall have everything you need for your everyday life, but you should break with your contacts so far. Secrecy is written by large.”

Ludvick thought quickly. He did not have any important personal binding. His job was mediocre, more than once unsatisfying. He worked a lot, but not on popular, easily promotable topics. He became a lone wolf. Perhaps it is time for a change.

“What shall I do if I am hired?”

“All documents are prepared. You sign your agreement here.” She conjured a paper out of the folder she was carrying with. Ludvick, enchanted, signed.

“Very well. Here are your flight tickets. The flight leaves tomorrow at 08:00 from Schwechat. At the arrival a driver will be waiting for you with a sign with your name on it. Go with him.”

Ludvick felt dizzy. “And what about the lunch?” run through his mind. But by then the woman has disappeared. Not even a scent of her perfume remained behind.

Ken

Ken woke up with a headache. And this was an understatement. The morning light caused pain, not only in the eyeballs but also in the neck; the otherwise lulling noise of street traffic from downstairs now felt as a fortissimo played by a whole symphonic orchestra, conducted by a capellmeister with latino temper. Hghh... He took a deep breath to relax the nerves. *Inputs... only one after the other. Please, I am human, after all.*

He managed to stand up and went to the bathroom. Being back he opened the fridge, took the orange juice bottle and drank. Little better. He cautiously opened his eyes. After a few seconds he could see shadows, a little later contours. The bed looked disordered, to say the least.

Then the blanket moved. And it talked. A girl's curly head appeared.

"Good morning, Ken. Did you sleep well?"

"Ah, ahmm. Good morning... uuh..."

"Come on. You did not forget my name!"

Ken tried to remember, but this was not the best moment for that.

"Bbbb..."

"Barbie. Actually Barbara Victoria Xena."

It cannot be! What a cliché: Ken and Barbie!

"Good morning Barbie. Excuse me for the mess. I suppose you have to do something today... somewhere else..."

Barbie made an offended face. Ken was used to it. This time was not the first time. He practiced his smile which used to attract women like a magnet.

"I am sorry, but I'll have to leave you soon. I must go to work."

In reality he did not have any program before noon that day. For that matter also not on other days. His habit was to be vigilant from 12 to 12, from noon to midnight. One thing was clear: he did not plan to spend the day with Barbie.

Ken sat at the counter and ate some leftover sandwiches. He dressed up and went to the door. He shouted back:

"You may leave any time ... before I'll be back."

He went to the garage and took his favourite bicycle. It was a special edition for city traffic, relatively speedy with five gears, wide, soft tires and a sophisticated damping system below the seat. Ken was an enthusiast of technics, he enjoyed the disassembly of machines, electronic devices, or any other construction he could touch with hands and feel the material tissue in it. He was teaching at a technical university; this day he was going to give an introductory course on informatics. The class was small, only around ten people. But this gave the possibility to go into details and discuss several examples.

After lunch he sat in his study room, shared with Bill, an elder colleague. This week Bill was away on a conference in Hawaii, with the topic of stochastic algorithms. He had the study room for himself. He switched on his screen and began to review his today's course. About the connection between entropy and information. In particular with the compression of information. His slides rolled on the screen.

→ information measure page 119

Suddenly a large window popped up covering his teaching plan. A small logo showing a coat of arms in red, green and deep golden colors blinked with a nerving pace. He felt anger. He clicked on it, and a video message started. A beautiful green grass bordered by oak trees and pines, a large fountain in the middle, clean roads covered by gravel in light beige to white color led on both sides towards an English countryside style castle. The camera zoomed to the terrace where a blond woman stand in a white costume. Light breeze played with her shoulder-long hair and she smiled uncovering teeth so white like in a toothpaste commercial. Ken smiled instinctively.

She waived and suddenly the picture became black. Light green letters occurred:

```
Ken Kenson, you are cordially invited to the
reception, given by his excellency the curator of
Our Future Society, tonight at Carlsborogh Castle.
Appearance informal.

Click on ''YES'' in the left bottom corner.
```

Ken did it, somewhat amused and delighted at the same time. Then some unreadable text was downloaded. He thought he saves this and deciphers later. But no. He was curious. He applied one of his favourite decipher app, called DEMENTOR, and after a while it produced the following text:

Date, time and address: Tonight 8:00 pm local time, Carslborogh Castle, Virginia Av. 1024, NC.

He could reach that party by car, it was only two dozen miles away. He smiled. *Finally, something happens, worthy of an adventurous mind, like mine!*

Ginny

It was 3 p.m., a foggy, not too warm not too cold afternoon. Time to time some raindrops fell, not all of them reaching the ground. Nevertheless the asphalt on the roads and pavements depicted a dark, shiny gray color, so characteristic for the Fall season in London. For that matter also for some other seasons there.

"How romantic!" thought Virginia. And at the same time it did not help her concentration to the talk at the afternoon seminar. Looking out of the window explored a huge grey block, not exactly homogeneous, in some patches lighter

than the others. The roofs and behind them the towers of the historical bridge could only be rather guessed than seen.

She forced herself to pay attention to the lecture. It was her will, to be proven in a profession traditionally held for men only, the informatics. Like if a woman would go with the hunters instead of the gatherers some 20.000 years ago. But she was convinced that the equal rights for women needed some proof of equal (or better) abilities. The proof was, however, not easy.

Men can think about unemotional, rational and very abstract problems for so long that by then a girl's brain would be draught dry. It is disputable whether this property is an advance or a drawback in the human evolution, survived only due to women choosing such men as partners, as gene donors for their children. *But why? What is the use of unemotional, abstract thinking? That they call "rational".*

The lecture was quite theoretical, the invited speaker arrived at the proof of Gibb's inequality. This was not new for Ginny – as she was called by this nickname by her close friends and some colleagues of equal rank. On the other hand she knew that it is only a special case of a more general property of the entropy formula: the overall convexity of the function defining the probability – entropy relation.

→ convexity of entropy page 120

She was nerved. Raising her hand she asked a question, wanting to provoke some break in the uniform grey mood of this event:

"Excuse me! What is this inequality needed for? Not to ask about the proof."

Some heads turned towards her. She enjoyed to be in the focus of attention. She shook her head, just a little, to show her shoulder-long mahogany colored hair to make smooth waves. Others in the audience were men. The lecturer halted.

"Well, uhmm, I suppose you know already this part?"

"Correct."

"Well I intended to give a more general introduction. But if you wish we can skip this part and go on with more recent results."

"Finally!" she thought. Loudly she told only, "... looking forward to it."

The lecturer presented the way to map non-additive quantities to an additive function of them; a general method beyond the logarithm.

→ formal logarithm page 121

“Is this solution unique?” asked a bespectacled colleague from the back row.

“Indeed, it can be multiplied by a factor, but the small argument limit fixes that freedom in the scale.”

The seminar went on in this style. Ginny was not satisfied with *l'art pour l'art* mathematics, she had a deep urge to see the purpose behind the formulas. What will it be good for? Not finding an immediate answer she had lost interest. The knowledge about the formulas, at least as a possible useful generalization if once upon a day she shall meet a problem, whose solution may require some elegance or degree of abstraction like she heard about today, perhaps then... she might remember.

At the end of the lecture she run out, not waiting for the obligatory polite questions aiming to illustrate interest in the topic and in the person. She went to the bathroom. *Oh my! That wears me out.*

She observed her face in the mirror. After washing with cold water the blood circulation restored her reddish teint. A small renewal of her make up seemed necessary. She took eye pencil, cream and a rouge out of her small red backpack and started to renovate her look. A combing of her half-long, thin hair came first and then the fine paintings around the eyelids. Arriving at the lipstick she suddenly realized that she is no more alone.

A middle-aged woman, dressed in a dark costume with some glittering inserts in the textile, appeared on her right.

“I hope I am not disturbing you,” said the woman with a musical voice.

She kept polite, “No, not at all. I was just sunken in my thoughts. Sorry. Here we are in a public bathroom.”

“You did not love the last seminar talk, did you?”

“I do not remember you in the room.”

“I was behind the scenes. As it is often our fate,‘ she sighed. ’Nonetheless I am interested in you, right now.”

Ginny made a surprised face.

“Well, I am straight.”

“No, not like that,‘ the woman smiled. ’My interest is of another nature.”

“What then?”

“I have a mandate to search for and hire first class scientists into a newly established research team. It is going to be a team work with mathematicians, informatics people, physicists and such. However, in a secret place, without contacts with the world known to you."

"That is secondary. What will be the research about?"

"A correct question. As far as I understood this myself, and you must consider that this is a very limited understanding, it will be a research about inequality. Not only in the mathematical sense."

"Interesting. But not yet fascinating. What do you offer?"

"Your transport, lodging and living will be cared for. You spend your time in a countryside castle with a great park, but you will not leave that area before the project is over. Upon success each participant will receive a bonus payment of about 25.000 dollars."

"Not bad. I am in the party."

"Very well." She took out a folder from her bag and handed it to Ginny.

"Here are your flying tickets, and a phone number you call upon arrival. Then you will get ground transport to the meeting place."

Ginny took the folder, opened it, and run through the content. Everything looked professional, with a logo in red, green and gold. *"OFS: Our Future Society"*. She took a deep breath and hoped that her trust in this sudden and somewhat strange offer will not be regrettable. She turned to say "thanks", but she saw only the swinging of the door.

Santino

It was a magnificent morning. 25 centigrades, a light breeze from the ocean, a few cumuli on the otherwise azure blue sky, and sunshine, sunshine, sunshine. Santino's working place in Rio was not far from Copacabana, actually 3 blocks away only. An ideal location for theorists, thinking day and night, discussing sometimes several hours long. The ease between hard thinking and mathematical exercises was close, so to say at hand.

Santino, somewhat aged, was a professor of theoretical physics, admired by the students, young senhores and senhoritas. Timeless young and sporty looking, in a great shape, moving and discussing with a latino temperament, and being at the same time jovial, good-willing and helpful towards young people wishing to learn. For the impertinents, on the other hand, he consequently showed a cold shoulder.

It was late morning, well, for people in Rio perhaps early morning. The discussion went around alternative formulas for entropy, when are they proper to use. An apprentice in physics asked about a certain formula, generalizing the classical Boltzmannian form:

"I accept to use a more general formula, whose particular case is the classical one. But I have difficulties in interpreting the generalization parameter. What makes this q different from the value 1?"

"Such questions are surprisingly hard. In the 19th century nobody could answer the question why the water boils exactly at 100 centigrades. Oh certainly, the scale was defined *by* the boiling of water, but other fluids boil at other temperatures. What explains the ratio of these boiling points?"

Realizing to be on a side track, what happened to him annoyingly often lately, Santino intelligently governed his speech back, towards the main message.

"We needed to understand the atomic structure of matter, the forces between atoms and measure their strength, and this was not enough. We needed to understand, how from such characteristics of the individuals emerges the behaviour of the complex whole."

"But water does not boil always at 100^o C. At lower pressure, say at high altitudes, it boils sooner."

"This is correct. But the boiling temperature can be calculated as a function of the pressure."

"The value 100 is not at all special. But the value $q = 1$ looks very special."

"Special? In what sense special?"

"Well, $q = 1$ for most physical systems."

"Most systems fulfilling a certain set of constraints. Raising one or more restrictions, it looks to me, even $q \neq 1$ systems may be described."

"Still, I cannot free myself from the wish of an explanation."

"Understood. Just do not forget: not having an immediate answer does not mean that there is no answer. Perhaps this will be more clear during your research studies."

"Perhaps."

Santino felt somewhat tired after these cocky questions. He raised from the table and decided to take a walk. He walked to an open terrace from where a view to the ocean was provided. He ordered a coffee, espresso art. It had an immediate effect on him. Nothing is better than café do brazil.

He took out his notes and started to review his next paper. This manuscript was nurtured since weeks and it was time to give it a better shape. How to present the q-entropy in an efficient and convincing way?

One possible way were to refer to the Hungarian mathematician, Alfréd Rényi.

Rényi constructed a general formula, the entropy expressed via a given power of the probability, and used a parameter named q. Alternatively one may start with the composition law not being additive and use q in the term deviating from the simple sum. The resulting formulas were not the same, but a given function of each other.

→ Rényi, Tsallis page 122

Santino sighed innerly. A little ambiguity here, it was always hard to accept. We need to resolve or explain this. A hard nut.

He raised, payed for the coffee, and took a walk along the shore. There was a paved promenade between the strand and the wide road with noisy traffic. Several kilometers long walks can be done here, Rio de Janeiro is a huge metropolis. Is it safe? Well, if you look like a local, who doesn't seem to be carrying anything more valuable than it'd worth the trouble to take away from you, you can safely walk during the day.

He needed some refreshing breeze, a little cooling of his head. Nearer to the ocean is always more movement in the air. Several other people enjoyed the nice day. The sandy beach was punctuated by little bars offering cocktails and some fast food, and covered by sunbathing men and women. A few palm trees donated shadow. In one of these shadowed areas someone had built sand sculptures; figuring exactly the young and taut bodies which were lying around, miles long, like a huge colony of marine mammals. The sand was formed hyper-realistic.

Beyond Copacabana the next beach started, called Ipanema. A little less frequented that day. Santino took off his shoes and socks, he waded bare foot into the cooling salty waves. That was good! Then he sat down at the beach and took again his notices.

Let us take under a magnifying glass the ambiguity. Perhaps these are only two equivalent ways to express the same nature.

→ q-log ambiguity page 123

Suddenly Santino felt a push on the back of his head. Then a kick and he found himself lying in the sand. He even did not have time yet to feel pain. He was angered.

"I have no money with me..." he tried to protect his head with the hands.

Rough laughter was the response. Several legs started to kick and tread, perhaps some fists also battered him. He began to feel dizzy and the breathing became heavy. Suddenly the pain disappeared and Santino lost his sense of being part of the real world. He remembered that resistance would only increase his pain – better to wait until the blind anger of his torturers flies away.

As from another universe he faintly realized the sound of sirens. A police officer occurred in his sight and slowly he was able to sit again.

“Who are you, sir?”

“My name is Wallis, Santino Wallis.”

“Can you prove this?”

Santino tried to remember of his pack, but he never carried ID card with him. At least this was a common habit of walkers at the beaches of Rio. He wondered.

“Please follow us, we shall clarify everything later.”

With some help of strong hands Santino managed to arrive at a middle-sized transporter in which he was seated and belted. The windows were darkened and the car had a noisy air conditioning system. Its effect was minor. They were driving with high speed. Santino wondered not hearing the siren tones. Perhaps his ear was damaged?

“Where are we going to?”

“You will be informed later.”

Any travel in Rio took its time. Wide boulevards or curved hilly roads, did not matter. Finally they arrived at a degraded small bureau, Santino was escorted inside and offered a seat on a pokey wooden chair without armrests. An unshaved officer, sweating, talked with a low voice.

“You, you say you are Santino Wallis? What are you doing in Rio?”

“I teach physics and do research.”

“At the Universidade? Will people know you there? It is far from the beach...”

“Today I had a lecture at a special place, not really far from the beach. I went there to think over some mathematical problems...”

“You may stop here. We take your fingerprints, and then verify your identity. So long be our guest,” and he pointed to the shabby lockup at the end of the room.

Santino did not feel to be invited. But what can one do? For a while better to sit in a limbo then be beaten on the beach. As long as you are alone.

Suddenly a tall man appeared in the door, in an elegant dark grey suit and a matching tie. On the jacket he wore a little emblem with red, gold and harsh green colors. He went directly to the commanding officer, showed him something. It made an immediate change. The officer straightened himself, clapped his ankles and waived towards Santino. His gesture meant something like *"He is yours"*.

This time Santino was driven in an elegant car with well-functioning air condition. Some drink was offered to him, he chose mineral water – to be on the safe side. They arrived soon at a luxurious looking villa, the car glided soundlessly to the main entrance of a two-storey white house with dozens of windows. He was escorted to a reception room. A jovial looking wealthy man in full white received him.

“Professor Wallis, finally I have found you! I am so happy and honoured to meet you!”

“Must I know you?” asked Santino disturbed. It was well possible that they met in the past, he just had forgotten. Neither the face nor the name wanted to be formed in his mind.

“No, not that I would know. But your fame precedes you.”

Strangely Santino did not feel flattered. “In what way can I be at any service for you?”

He meant his question ironic. But the man took it as most natural. He must have been used to the fact that other persons do services for him.

“Professor Wallis, please, escort me into my study. I have a proposal to discuss with you.”

They went to a small study room, equipped with comfortable chairs, a working desk, a big terminal and keyboard, and possibly several more hidden infocommunication tools of the day’s fashion. Santino felt angered because of the events at the beach. He started belligerent:

“Was it necessary to get me beated? Not a good entry for a proposal.”

“I beg your pardon, what are you referring at?”

Santino briefly explained what happened with him at the beach. The man wondered.

“I assure you, Professor, we have nothing to do with that. An unfortunate case, which should be filed to the local police.”

Perhaps it was true. Factually. Santino calmed somewhat.

“What is the essence of your proposal then?”

“I have the mandate to offer you a leading research position commensurate to your expertise. It is, however, abroad, and my entrusters wish to keep the results secret, before they decide whether it is something for which the general public is already ripe enough.”

"This sounds mysterious. But please, go on."

"Your transport and livings there will be cared for. Once you sign the agreement we hand you your travel documents and some pocket money out. About the rest you will be informed on the spot."

"Sounds fine. When shall I start?"

"Your flight leaves tomorrow at 12:00, local time."

Santino did not believe himself, but he signed. Perhaps the events of the day moved him emotionally towards to go away from Rio, to some less crowded place. He felt he could need some contemplation in noiseless peace.

Vilmano

The construct was much bigger than one would think. A nicely shaped dark brown wooden slope of about ten-twelve feet long and five feet high. On the top a finely carved and polished channel and, the most unique feature, small gates over the channel, in increasing distances. *"What an ingenious solution!"* thought Vilmano. And that was the point: not having any reliable device to measure time, no chronometers, only a pendulum or one's own artery pulses to count heartbeats, how to find out the very nature of movements.

That was what Galileo had done. First the discussion was about falling bodies, how they accelerate during their fall. That was not available for the naked eyes observation, Galileo had found a way to slow the motion down by using slopes. But still, while marbles or wooden spheres rolled down on a slope slower, the answer to the question how much space they left behind in a given time remained unclear. Well, speculations had always been around about this, but no demonstration, no measurement before.

Experiments about movement always need measurements of time. The finer the better. In a 19th century wooden slope device small gates are erected along the groove on the top, with small bells. Each time a marble passes a gate the corresponding bell rings. The timing of consecutive bell-ringing could be followed by the ear. One adjusted the position of the gates until the ringings followed each other in an equal pace. Tinkle, tinkle, tinkle, tinkle. And, see wonder, then the distances between the little gates were as $1:3:5:7$. Oh, if the distances increase as odd numbers, then the positions follow a series of quadrats: $1\times 1-0=1$, $2\times 2-1\times 1=4-1=3$, $3\times 3-2\times 2=9-4=5$, $4\times 4-3\times 3=16-9=7$. Expressed by a general formula, $(n+1)^2-n^2=2n+1$.

Vilmano was fascinated by old devices, let they be for physics, chemistry, astronomy... He remembered his high school physics course with warm feelings. It was exciting to intrude into a world of wonders, surprising factoids and theories about how the universe functions in its smallest and largest parts. But he did not follow this inclination towards philosophical questions. He studied economics, business and management. Here numbers still play a role, but mathematical models, in particular methods of higher mathematics, were not in fashion during his university studies. Not yet.

Economics wants to be useful, but economic theories are rarely so exact, that the predictions will be *quantitatively* verified. One absolves the studium in order to build a network of good relations with important people. Not that it would be impossible to have theories, expressed in math formulas, about social and economic processes. At least about parts of them.

Vilmano left the Galilei museum. It was arranged in an old house near to the bank of the Arno. This was the downtown of Florence, a place where the ghost of Renaissance even to date whirled around in the air, as a superfine scent beyond all other natural ones. The famous bridge of the gold merchant guild, Ponte Vecchio, was in sight.

This time he decided not to go over the crowded bridge. Tourists everywhere. He turned towards the dome, getting a superficial glance of the David sculpture at the Uffici corner (the original one was inside and showed for money only). He needed a cup of cappuccino to start the day. He sat down at a small table at a coffee house and took his laptop. He wanted to have a final look over his slides prepared for a talk for students.

He changed his order from capuccino to an americano; a thin coffee in a big mug. He has got a special mug, with a sign 80/20 pressed on it. He smiled. The famous Pareto law: 20 % of the population owns 80 % of the total wealth. But how did it come? How is wealth distributed? And what determines this or another distributions?

→ Pareto's 80/20 rule, page 124

A teenage looking girl, dressed as an American tourist, was looking over to Vilmano's table.

"What are you doing?"

Vilmano looked up. His surprise must have been mirrored on his face, since the girl laughed. Not in an evil way, just as expressing a natural happiness.

"Excuse me if I disturb you. But you looked so serious."

Vilmano grinned. He talked to strangers willingly. In particular, to young girls.

"Oh I am calculating income."

"So much do you have?"

“No. It is about all incomes of all people. A problem of academic interest.”

“For me income is not a problem. Someone will always have enough to support others.”

“I am afraid it is more complex than that. Do you know that it was an Italian who first suggested a formula for the wealth distribution in the US?”

“Indeed?”

“His name was Vilfredo Pareto. He died in 1920, but his formula fits also today.”

“So what? It is not easy to invent some formulas? For what purpose, anyway?”

“Formulas which describe the truth – not at all easy. And once we know how many are rich in a society, we may know our blind chances to be rich.”

“I shall marry a rich man, for sure. For that I do not have to know math.”

Vilmano shrugged. He returned to his thoughts, and took a serviette to note some of the formulas, which did not tend to leave his mind in peace.

→ Pareto distribution, page 124

Vilmano was surprised. According to the classical 80/20 rule twenty percent of the population owns eighty percent of the wealth. Interestingly, to that wealthy quintile fraction belonged also people earning less than the average. Down to eighty seven percent. A strange conclusion.

He checked for other Pareto rules: some gave a splitting income value over, some under the average. A 90/10 rule – some were preferring – meant that from about 66 % of the average income one belonged to the 10 % who is earning more than 90 % of the total. The natural splitting value exactly at the average, assuming Pareto distribution for the PDF, was for the 75/25 rule. The softest possible rule, 68/32, rendered only people with 115 % of the average income or more to the rich set. *So perhaps the* 80/20 *rule was not so dramatic at all!* Of course, looking the other way, 80 % of the population earned less than the 87 % of the average. But the really rich people make out much less than 20 % of the population. For the 80/20 rule the fraction of people earning more than ten times the average is about one single percent.

Suddenly the girl from the tête-à-tête before returned. But she was not alone, a sober dressed woman accompanied her. The elder woman looked ageless, like a secretary, pardon me, personal assistant of some very important and influencer type macho boss. She came to Vilmano with energetic steps and a determined face. *“Gosh! I hope I did not commit anything wrong or illegal by talking to the girl...”* came the thought suddenly across his mind.

"Dr Zotti? Vilmano Zotti?" asked the woman resolutely.

"Well, hm, yes. How did you guess?"

Instead of answering the woman gave him a cover, dropped it to the table, and it landed on the top of the 80/20 mug. Vilmano did not move, continued to stare at the woman. The two turned away and disappeared in the labyrinth of the narrow and curved streets of the old city.

Vilmano, absent-minded, put the cover into his pocket and forgot it for the rest of the day. Returning home after a filled evening with discussions and phantasy trips, it fell out from his jacket. He took it and opened the envelope. A letter was inside, written with carefully calligraphed letters, as follows:

Dear Dr Zotti,

We would like to make you an offer: we invite you to an adventure you have never had before. We would like you to join to an international research team on social and economical inequality, its origin and dynamics. We invite you to our start-up meeting at Carlsborogh Castle, Friday, next week.

You can go to the OFS shelter at the airport and only prove your identity by your card. Your travel (flight, ground transportation etc.) will then be cared for.

Looking forward to meet you at the party and to welcome you in the Inequality Research Team afterwards.

Sincerely

Ego Zeus Georgos
Our Future Society

Vilmano shrugged and threw the letter onto his disordered desk. Then went to sleep. Still, on Friday he found himself on a flight westwards.

Chapter 2

Stories and Histories

Party

"How fortunate a day!" thought Ego Zeus. As a late reminder to the Indian Summer the sky was deep blue and the sunset painted the bushes and trees in a positive, warm light. It will be dark soon. Ego, dressed in a cosy white suit made of fine cotton, stood on the terrace of Calsborogh Castle, near to the main staircase leading to the entrance. His secretary, a beautiful woman in her forties with blond hair, close to him enjoyed the warmth of the sunset, too. They were waiting for the guests invited for tonight, with a huge tray of welcome drinks arranged on a small table with wheels.

The first limo crossed the parkway and stopped softly on the stony road in front of the terrace. A genuine Italian, dressed in a sporty cloth oriented on the fashion of the 1920-s, jumped out of the car.

"You must be Dr Zotti," greeted him Ego with a bright smile.

"You bet,' replied Vilmano somewhat too short. 'Where is the party?"

"Since you arrived first, not much is going on, yet. Please take a welcome drink: Martini, Scotch, Orange Juice?"

"Martini, of course. Please, call me Vilmano. May I join you in welcoming the other guests here?"

"Yes, why not."

As Ego stepped a little back allowing Vilmano to join them on the terrace, and revealed Ms Kunz this way, an even brighter smile appeared on Vilmano's face. *Ollalah!*

"Lady?"

“You shall call me Ms Kunz. And no ground for any special friendliness from my side.”

Vilmano bowed. “Then we shall restrict to *my* friendliness.”

All turned towards the park, it was sinking slowly in the darkening twilight. Ego pushed some buttons on his minimal cell phone in the pocket and the lamps, equipped with warm orange LED lights, started to brighten the way up. The next car was driven up the way.

Two guests have arrived. First a tall, well dressed and jovial looking man jumped out revealing his southern temperament. He opened the door and kept it for the other passenger, a young woman with reddish-brown hair. She stepped out not taking the hand of the gentleman, throwing an icy look instead. She frowned, as someone feeling chilled.

“Welcome, Professor Wallis, a special welcome, Dr Astra!” Ego meanwhile checked the photos of the invited guests from the internet. It was no magic to identify them.

“Good evening! Hello!” told Santino and Virginia unison.

“Please step inside, we have prepared some drinks and food. Vilmano, would you please show the way to our new guests?”

Vilmano shrugged and smiled. Then he took the direction to the entrance door. The three entered the castle. The site was really impressive. A wide staircase in the middle led upwards, it was covered by a dark red carpet, the banisters were nicely carved from mahagoni wood. A huge luster was fixed high up at the ceiling, and also some chandeliers were on the walls contributed to a friendly brightness inside. High and thick wooden doors to the left and to the right. The dining room must have been to the left as faint noises and sweet odours witnessed it. The three walked over there.

Meanwhile another car had arrived. A yellow cab, seemingly from the airport. A young looking man in his best years got out and approached directly the terrace. A quick check on the list confessed that it was Ludvick Holzmann.

“Gooood eevening!” he said in a Viennese accent. Ego greeted him.

“Good evening, Dr Holzmann. I suppose you have already met Ms Kunz,” he waved his hand towards his secretary.

“Oh yes. She owns me a lunch for two,‘ tried Ludvick diffidently. ’At least that was the way she approached me, if I am not mistaken.”

“Do not worry, Dr. Holzmann. The evening will be long today,” japed Ms Kunz with him.

“Please choose your welcome drink and join the others in the dining salon,” suggested Ego nonchalantly. Ludvick took it literally: moped up an orange juice and then almost run into the house.

“We have already four from the seven scientists you invited,‘ ascertained Ms Kunz, ’but what about the rest?”

“We are waiting for the American guy, still. Two others will not come in the conventional way.”

Exactly by ending this sentence an old car appeared with dimmed headlights and a loud clink-clank. It stopped at the terrace and the engine made a noise which sounded final. Ken shrugged and jumped out from his old buggy.

“Hi! Is here Carlsborogh? I thought I was invited to a *party*.”

“Indeed. Please take your welcome scotch,” grinned Ego. Ken took it and swallowed it in a second. The effect came one more second later; his face became red, his eyes narrowed, “Boo, it was good!”

Ego invited all to go in, and they did proceed. Ms Kunz ahead, as the courtesy demanded it, then the young guest, closing by Ego, the husband. Pardon, landlord.

In the dining room the guests already started to get social. Costing one or another appetizer, sipping on their drinks they looked as well entertained by each other. Ego went to the end of the long table, prepared for a dozen of participants, and announced aloud:

“My lady, and gentlemen! Please be seated, we would like to start the evening program.‘ While the five guests got seated, Ego looked around at their faces. He was satisfied, for everyone looked relaxed. ’First of all welcome at Carlsborogh Castle. In the name of Our Future Society Foundation, the OFS, I ask you to take some refreshment and food before we start the hard work you came for.”

Suddenly some brattle was heard from the antechamber. Two more persons marched into the dining room, Fred and Rudolph. Ego smiled. *Finally, the circle is completed!* He invited the last two with a wide gesture to find their seats at the table.

“Once again I welcome you all. Thank you for accepting the invitation to this extraordinary research commitment.‘ He paused for a few seconds until the silence has grown to an optimal level to bed for a maximal concentration. ’This evening is for leisure, but I suggest to start getting acquainted with each other.”

Some bowed, some murmured kind of *yes* or *indeed, it’s high time.* Although the environment was relaxing, in some aspects directly luxurious, the society of unknown people still was an obstacle in a way. It hindered the real relaxation of nerves, which was needed by now for all invitees. Ego continued:

“I suggest that everyone tells a few words about herself or himself. Let us start with the names, and then telling a favourite year and the closest person to a scientific idol. For example, my name is Ego, Ego Zeus Georgos, my favourite year is the first year –1 AD–, and my science idol is Newton.”

The others reacted in a mixed manner. Some made a mouth, some nodded, others smiled as being pleased by the game. Then slowly, after each other, they started to contribute.

“My name is Ken, Ken Kenson, my favourite year is 1968, and the idol is Claude Shannon.”

“Santino, Santino Wallis, favourite year 1989, and my most esteemed scientist is, say, Constantino Tsallis.”

“Vilmano Zotti, I am an economist, favourite year 1929, scientist Corrado Gini.”

“Fred Serényi, 1956, Alfréd Rényi.”

“Rudolph Genius, 1888 and Rudolf Clausius. Not because he was also Rudolph.”

“Ludvick Holzmann is my name. I loved the year 1999 with the stupid panic about the Y2K effect. The most important scientist in my eyes was Ludwig Boltzmann.”

“And last, but not least I am Virginia Astra, doctor of mathematics and informatics. I have no favourite year, but yes, perhaps 2012. My favourite person is Elizabeth Tudor, the first queen on the English throne, yet a favourite scientist... maybe Ada Lovelace.”

“I have another important question,‘ said Fred, ’why did we accept this invitation?”

The society at the table became louder. Several answers crossed each other. *Curiosity* and *change* and *an interesting offer* were the most frequent phrases. Ego smiled wisely. *It is useful if they reduce their stress.* He stood up to be heard better.

“Dear Professors and Doctors! We shall have ample time for detailed discussions in the coming days. However, let me give you a first sketch of the problem which brought us together.”

He waited for the silence which occurred in a minute. A minute of sixty seconds sometimes feels really long. Ego started after a deep breath, he dimmed the room and switched on the projector hidden in the ceiling. His first slide appeared.

SWOT analysis: the state of humanity

- Strength: mankind has survived several generations, population is increasing – more than ever.
- Weakness: increasing inequality, stress, aggression, casualties.
- Opportunity: more understanding of ourselves and the rules of the universe → more power to change.
- Threats: it may be too late to correct the rules by now.

Before Ego could start to expound on this slide, the scientists started to comment. In fact, it occurred like an immediate immune reaction – none could control the habit of showing that he is smarter than the lecturer. And in fact – deep in their hearts – they were all convinced that it is really so.

"A short summary of our history! Extremely naïve," Rudolph remarked.

"'Several' is a qualitative term. We cannot build on it," sniffed Ginny at it.

"'Increase' is a global statement. We have local counterexamples," so Ken.

"Knowledge is power. Nothing new about it!" inserted Fred into the turmoil discussion.

"I am afraid we will not be able to 'correct the rules of the universe'," alleged Santino.

"You are crudely oversimplifying..." reacted Vilmano. Only Ludvick kept silent.

Ego first wanted to react. He felt to be provoked – with right. But quarrel was not on the agenda this evening. Instead he smiled and went on to the second slide.

"The main point is to understand 'inequality'. Why does it occur, what is its mathematical nature and what are the possible ways to manipulate it. In particular, the economic inequality, which seems to be the base for social, ethnical and other types of inequality, too."

Questions

1. How to measure inequality?
2. How much inequality shall we expect?
3. Is there any natural trend behind its increase?
4. Can we reverse such trends? On what price?

Ludvick was making notes. He wrote in a little notebook he always carried with. Do we see trends because they are really there, or just because of the way we are calculating?

"I have a nice example. Say someone tells you to increase your salary with a given percent, and then they reduce it with the same percent. Do you think you are even?"

"I bet not!' said Ginny warmly. 'Otherwise you would not have brought this up."

"Right. See first the general rule:"

→ non-additive percents, page 126

"The surprising fact is, that a composite effect of two, virtually balanced action is always a loss. It has a definite sign."

"Wow!' remarked Vilmano. 'You are dashing to the heart of the banking guild."

"Is it more than trivial algebra?" asked Ken.

Ludvick tried to explain. He felt *deja vu.* He was after something deep and important, but his audience degraded it.

"Of course, more! We have a rule, a procedure with defined formulas, from which you do not guess from the beginning that it can have a one sided result. It is like the entropy. As Boltzmann's H-theorem, symmetric rules lead to an asymmetric result."

"Hmm. Is it not some convexity in your formula?" inserted Santino an inquiry.

"I just wanted to claim that some one-sided behaviour of formulas may stem from the very definitions, from our way of viewing, not from the real nature of the problem."

"We are getting tired, guys,‘ noted Ken. 'Look, Vilmano has found a piano."

All looked there, and indeed Vilmano sat and started to play an Italian *canzone*. The others moved closer, amusing and drinking for a good while. They sang quite a number of songs – musical or not. Well, Italians cannot end an evening without singing.

Night Alarm

After midnight a peaceful silence settled on Carlsborogh Castle. Seemingly all guests retired to their separate chambers. The artificial lights were all dimmed and only the half-moon shed some silvery shadows onto the park and the house in its middle. The guests, the landlord, and his personal assistant, Ms Kunz, were sleeping deeper and deeper.

Around 3:00 in the early morning a little buzz sounded from the cellar. Then the alarm lights were ignited and a siren started to repeat its signal on a nerving tone. All relaxed dreams must have had an abrupt end. Sleepy faces occurred in the doors and after some disorientation the people gathered at the top of the staircase. The sound seemed to come from below.

"What the heck should *that* be? Are we under attack?"

"Or a fly triggered the alarm device?"

"Is it my car? No, thank god, it is not."

"We have to check the cause," stated Ego calmly and marched downstairs. The others followed him. Scientists *are* curious people.

In the antechamber, beside the old knights' suit of armor, Ego touched a certain point on the wall. His fingerprints were read and a hidden door slid away. They entered to the dark staircase leading down. Ken switched the light of his mobile device on, and then two others did so, too. Ego resolutely marched further on the lead. The alarm siren became louder, already hurting the ears.

They arrived to another door, an entrance to an underground laboratory. Ego opened it, and the mass behind him urged to enter. But then all stopped, frightened. A body was lying on the floor, just in front of the toroidal device, huge enough to incorporate the whole team.

"This is the device I have entered," noticed Fred in a low voice.

"And this is the cellar we first met," added Rudolph.

Ego stepped closer to the lying body. Then he pushed some code in his personal pocket device and finally the sirens were muted. The sudden dummy silence felt strange.

“Does anyone knows this person?” he asked.

The answer was a complete silence.

“I think we have to call the police. Ms Kunz?”

“I have done it, already,” said Ms Kunz in a matter of fact tone, demonstrating that a good secretary knows how to work pre-emptively for the boss.

“While waiting, we may entertain ourselves with telling stories,‘ suggested Rudolph. ’It is a classical method to reduce fear and confusion.”

“Then you should start, you psycho man!” hissed Ginny.

“Very well. My story is about the fate of the Universe. As we know entropy is spontaneously growing in any energetically closed system. This reflects the trend of equilibrating the temperature between bodies. At the end all bodies must have the same temperature. And then every energy exchange stops, no more work can be done. Physically nothing changes any more. The heat death of the universe establishes and it stays forever.”

“What a wonderful story!‘ clapped Ludvick ironically. ’But what if the universe is not closed? Just a part of a bigger universe, which is still a part of something even bigger, ad infinitum? Then the life before the heat death can be infinitely long.”

“Or what if the trends we observe to date will reverse in the future? The present arrow of time in our expanding universe will change to its opposite in a collapsing universe?” inserted Fred.

“A collapsing universe may reach its end sooner than the heat death,” remarked Santino dryly.

“Then suppose an infinite series of expansions and collapses. An oscillating world,” defended Fred his suggestion.

The dispute was abruptly ended by a new noise from outside. A helicopter landed in the park and several heavy steps were heard. Two dressed in civil, and several uniformed policemen appeared in the lab.

“I am Inspector Holmes,‘ introduced himself one of them briefly. ’What do we have here?”

“Good morning, Inspector. It looks like an accident with our experimental device,” answered Ego.

“Is he one of your colleagues? Have you touched or moved anything?”

“No and no. I have just cancelled the alarm, when we entered.”

“Doctor, will you please examine?” turned the Inspector to the other man dressed in civil. The doctor took his gloves and a suitcase for medical emergency and checked the body.

“He is still alive. Must be brought to a hospital.”

The Inspector beckoned and four policemen brought the injured out of the house on an improvised barrow.

“We shall take your personal data and you must not leave this house without getting an official allowance from us.”

“Yes, Inspector, we will not.” Ego and Ms Kunz escorted the police out of the castle and the guests, slowed down and fatigued, plodded back to the dining room. None could sleep anymore and they needed some coffee.

“What is this strange device downstairs?” inquired Ginny.

“Don’t know. I entered in it in Budapest and was waked up here,” replied Fred.

“I also woke up here in that device. And I’ve met Fred in the lab,” added Rudolph.

“We must interrogate Ego about this machine,” suggested Santino.

“Interrogate? Don’t you misjudge the power relations here?” inserted Vilmano.

“In any case I would love to study it closer,” sighed Ken.

“And who is the man found there laying?” drilled Ginny deeper. A general silence followed. Finally, Fred made a joke:

“I bet our ignorance has maximal entropy on this issue by now.”

“A little less than that,‘ added Ken, ’since we already know that he cannot be any of us. I hope.”

Some frightened look was the consequence. A bad joke.

“If this were a time machine – purely hypothetically – then in fact the body can well be one of us,” remarked Rudolph.

“And if this machine would slide between parallel universes, it can be a close version of us,” fetched Ludvick the word in a low tone.

In the view of these perspectives our scientists decided against it. An explanation was postponed. And none talked about it further.

History or histories

After a tasty breakfast with ham and eggs, and coffee with fresh bakery products the team gathered in a working classroom on the second floor. Seats and tables arranged in a semicircle, with some fractal shifts relaxing the order, seated the participants in pairs. In the focus there was a special, bigger table with monitors and keyboards, and equipped with a comfortable looking chief's chair. Ego took that place and opened the session.

“Welcome dear colleagues to our first working session. Since my research goal is to learn more about inequalities, their origin and dynamics, a first natural study object is offered for this by the human history. Is it possible to attach any mathematical models describing this? Does it have laws, similar to or different from the natural laws formulated in e.g. the Newtonian mechanics?”

“A good question!” reacted Rudolph.

“Normally a law and its math model describes real world phenomena which can be repeated...” added Ginny. “Human history does not seem to be such a case.”

“Statistics is needed in observation,‘ continued Ludvick, ’otherwise not even a probability can be probed by true facts.”

“On the other hand we do make models for unique objects in change, too,‘ turned Fred on the discussion. ’Examples are cosmology or chances of life on an exoplanet. We apply the Einstein equation to describe the behaviour of space and time on a large scale in our universe.”

“Please, let us have a closer look on statistical physics first,‘ interrupted Santino. ’We describe average properties without knowing the fate of a single atom. We just assume that we have many clones of the same objects which do not differ.”

“And don't correlate with each other, either,” inserted Vilmano.

“Yes,‘ continued Santino, ’we make a statistics over independent and identically distributed random copies at the base. Only some, in fact just a few, macroscopic constraints vary the picture somewhat.”

“Can we apply a similar approach to human history?” asked Ego.

“On this planet we had and are having only one history,‘ remarked Ken, ’but it has parts, and in the past even had ’independent’ parts. Do they show regularities, any recurring pattern?”

“If there were simple models, we could program a simulation. A kind of history game,” suggested Ginny.

“Such games exist. My favourite is Sid Meier's *Civilization*,” replied Fred.

“Then what about making statistics on games?‘ followed Ginny this thread.

'That could be done in less than a few thousand years..."

"I did it,' announced Fred. 'I have constructed an oversimplified model of a few equations and then checked it with results of several dozens or hundreds of games. Also with some UN statistics about the real mankind. I could lecture on that."

"Please do!" proposed Ego and invited Fred to the traditional blackboard with chalks.

Fred walked out and started his talk.

"Fine. In my view the most important variable, which characterizes a unit of human societies, must be selected first. I take for it the number of persons, the 'population', $p(t)$. I call the unit of study a *civilization*, a structured set of people living and working together, following common ideals and goals, distributing resources among themselves and reacting as a whole to challenges from other units. My primary directive is to model its change in time. I show you the basic equation."

→ civilization mean field, page 126

"Your quadratic law assumes that all married couples met randomly," noticed Ken.

"Correct. I think this is a defendable assumption, by large,' reacted Fred. 'Let me continue with other effects now, later we may generalize to other powers in the population growth law, beyond 2."

→ Malthus, page 127

"As I have shown you now, in some cases the world population would grow without limit, it would diverge."

"But experience does not show divergence!' exclaimed Vilmano. 'I admit sometimes I feel that too many people were around... but not *infinitely many*."

"Exactly. So we have to improve on this model. There must be another factor which saves us from the population singularity," explained Fred patiently.

"Wow! A 'no naked singularity' theorem for the human history!" jubilated Ludvick with a wee of irony.

"I would not go so far,' remarked Rudolph, 'let us face the more realistic version of Fred's model first."

"Thanks for this suggestion!' bowed Fred, and continued his lecturing. 'The sensitive improvement is allowing for a drop in the fertility rate, a factor which reduces the number of children per couple."

→ saturated demography, page 127

"We have lost now the dying out at low population, and gained a saturation at a high population instead of the singularity," noticed Ego.

"With the caveat that now all factors can be negative, so at a too low resource area or too high mortality rate the saturation number, p_S, itself can be negative. These are the dying civilizations we had before," answered Fred.

"Another question: What shall we do with phenomena as colonization, wars, trade, discovery – which all escorted human history?" pushed Santino the discussion in a new direction.

"Ah,' sighed Fred, 'we cannot have everything in a math model. However, some of these effects modify the controlled area A on the short term, and then even the remaining parameters can have an evolution, or some statistical fluctuations."

"If we want to describe everything, we describe nothing,' added Vilmano. 'Let us see Fred's further assumptions."

Fred cleared the blackboard and took a deep breath. He was moving on more and more swampy ground.

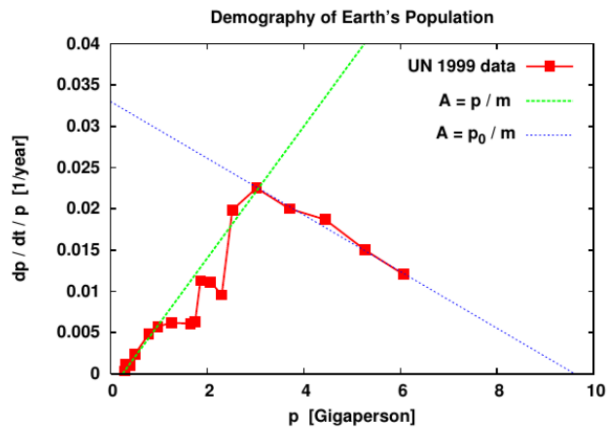
→ resource dynamics, page 128

"In this more elaborated model the population growth depends on the area from which the necessary resources are gathered. It may grow with growing population first, but it saturates when the world becomes 'small', and the resource areas of civilizations collide. The limits of the growth can be very real."

"So we have now a complex simple model,' remarked Ken dryly. 'But what are the true facts?"

"Yes, are there any real world data on the population number? Moreover, *historical* data?" inquired Vilmano.

"I plotted UN census data, supplemented by some historical back-estimates. That I have found on the internet. In this figure they are plotted." And Fred sent from his mobile a picture of it to Ego, who displayed it in large for all.



The annual relative growth rate, $\dot{p}/p$, versus the population, p. Based on UN data from 1999, and historical estimates.

"Here $\dot{p}/p$, the annual relative growth of Earth's population is plotted against the actual number, p. On this phase space plot a rising and a falling linear piece can be identified," Fred explained.

"Amazing! What?! Unbelievable!" the mixed reactions came.

"We had big crises, several times. Plagues, economic crashes, world wars. What about them?" asked Vilmano.

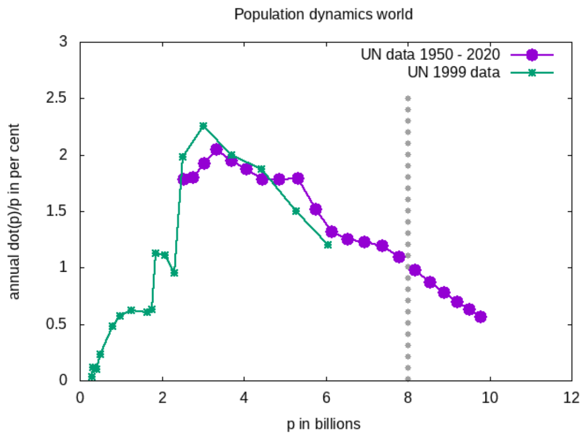
"Indeed there are two parts when a backfall from the rising linear can be seen. Those are the two world wars."

"What, so recent? At which year your data start at all?"

"I think somewhere with one-hundred million, around AD 1. Of course, this is only an estimate. The seeming growth started in the 17th century."

"And are we over the maximum point?" asked Ginny.

"According to the data I had,' explicated Fred, 'somewhere close to 1960. There are also newer data, until 2020," he projected a new figure.



Annual growth percentage trends versus the total population of the world. Above 8 billion only predicted points by continuing the linear trend from the last two data points.

"They differ!' exclaimed Ludvick. 'Only a general trend is similar. However, earlier predictions are falsified."

"Indeed. There are also methodological differences between the green and the violet data: the change in the population was estimated by a two-point difference formula earlier, while the new data are summaries of annual growth percent. The prediction is simply continuing the linear trend, it may well be false again."

"So shall we talk about history or about histories?" put Santino a new question.

Ego tried to summarize:

"I think all histories, characterized by a growth, based on resource usage, which as a back-reaction can become tight, will show such eras of early and late developments. All other details may or must vary."

Then a nice musical tone sounded, alike an Asian gong escorted by some harp tunes in minor. Ms Kunz's smiling face appeared on the main display.

"Hello everyone! The lunch is ready in the dining room."

Chapter 3

Information Investigation

Codes and Entropy

The lunch time was passed in a relaxed manner, the scientists talked in smaller groups of two and three. After having cakes and coffee they returned to the classroom to continue the disputation. Ego started the session.

“Thanks Fred for your presentation. But please do not forget that our research goal is to constitute a theory about wealth inequality.”

“I would be more interested in the machine downstairs,” claimed Ken.

“Yes, we too,” sounded from several mouths.

“Hmm,‘ contemplated Ego, ’I do not know myself all details of that. This is a machine for alternatives.”

“Alternative whats? Universes?” asked Rudolph.

“It occurs to me that it simulates alternative realities. The simulations feel real, quite a lot.”

“Can it be used for travel, too?” inquired Fred.

“In a sense, yes.”

“Are we really here now, or just are connected in a virtual reality simulation?” inserted Ginny her question.

“Can you distinguish?” replied Ego.

“If you can select differing universes, that also means, that different space and time locations can be selected, too,” reflected on the possibilities Santino.

“Past and future are different. Even in a simulation,‘ interjected Ludvig. ’The entropy growth gives time a direction.”

“Maxwell’s demon,‘ injected Fred, ’will be hot by selecting the atoms, when he tries to shuffle energy from a cold body to a hotter one. Turning bits must be a physical work and all physical work must be escorted by heat production. The total balance is more entropy in the total system.”

“The reach of a maximal entropy state in a closed system is a universal trend in nature. If the total universe has no connection to give heat away and cool down, then the so called ’heat-death’ is unavoidable,” contemplated Rudolph.

“So we conclude that a universe-selecting machine, in the sense of Everett’s multiworld theory for elementary, quantum level measurements, must cost energy and some entropy increase. Or equivalently information loss, since at maximal entropy every microstate has the same probability, and therefore our information about it is minimal,” summarized Ken.

“But, specifically, what kind of information is lost when going from one universe to another?‘ asked Ludvick. ’What if we exchange the starting and arriving universes? Sliding to and back is a double increase in entropy?”

Ego wanted to halt this discussion, it leads far away from the research goal in a fractional space of possible questions. “Why is information compression an entropy loss?” he wanted to know.

“Well, in case of a lossless compression, there is no information loss, consequently no entropy production. Any series of code fragments can be converted back to the original strings, uniquely,‘ tried Ginny to explain. ’It is compressed only because for more frequent elementary code combinations one uses shorter coded words.”

She put some formulas onto the screen.

→ text code length, page 129

“And why is this coding unique?” asked Ego.

“Because the basic 2-letter codings were chosen as powers of 2. That makes by decoding to count only that how many zeros follow a bit 1 ,” explained Ginny.

“And now comes the important statement by my hero, Claude Shannon,‘ inserted Ken with a confident smile, ’the best coding *must* use short codes for the most common occurences and the longer ones, the higher binary values, for the most seldom combinations.”

He jumped to the whiteboard and drew his favourite formulas. “I want to give you an intuitive picture,” was his starting remark.

→ codelength entropy, page 129

“And this is the Shannon entropy,” finished Ken with a victorious smile.

“It is extensive, is not it?” inquired Santino.

“Indeed. That even I can follow,” said Vilmano and wrote his chain of thoughts onto the whiteboard:

→ additive information, page 130

“The interesting question is, when does the coding become non-extensive.” drilled Santino further.

Ginny shrugged. “Surely, if there is correlation between the segments. For example when after AA an AB is more probable than a BA or BB. With other words the joint probability for a quadruple is not a product of those ones for the couples.”

“In English the ’T’ is the second most occurring letter, after ’E’. On the other hand ’X’ is quite seldom, fourth from the rear end. Yet, the couple ’XT’ appears in some – not very rare – words, like ’next’, ’dexterity’, ’extra’, ’extreme’ or ’baxter’[1]. Meanwhile I cannot cite a word with ’TX’,” interjected Fred. He projected the table he found on the internet.

→ English letter frequencies, page 130

“I inserted lines at the cumulative probabilities of 50, 70, 80, 90 and 99.5 percent.”

“It is fun!‘ exclaimed Ken. ’So here is my message at the thirty percent level: ’E’.” He waited for 2 seconds to increase the effect. Then continued:

“At 50 % my message is ’I OE O’. Is it not much better? And it stays the same until the 70 % level...”

“For guessing we have to go to the 80 % level: ’I LOE OU’.” Ken was fixing his look on Ginny, amused.

“And what is the level of getting the total message?” asked Ginny, shaking her head.

“Well, in this case, the 99.5 % level ensures first ’I LOVE YOU’,” concluded Ken practising his usual magnetic smile. Seemingly no effect on Ginny, she even did not show a sign of a faint disturbance. Her eyes appeared sparking from anger.

“Big deal! Short texts will not reveal the actual statistics,” criticized Fred.

[1]The name Baxter means baker, a middle English version of it was *backstere*.

“As if you wanted to apply thermodynamics for a dozen of atoms!” added Ludvick.

“So according to your table the code of an English text should use 1 for E, 10 for T, and so on?” wanted Ego to arrive at a conclusion of the afternoon meeting.

“Exactly! Yes!” told Ken and Ginny simoultenously.

“Not quite,‘ noticed Fred in a lower voice. ’The space character also has to be coded. And it must have been the most frequent one, since the total number of letters from this table is 183.139 in 40.000 words. Meaning about 40.000 spaces if this would be a unified text. This is more than the 21.912 occurrences of ’E’. So the best code uses the binary 1_2 for the space, 10_2 for ’E’, 100_2 for ’T’, and so on.”

“Eh, good. And then the code length per coded symbol is optimal, approaching the entropy formula by Shannon? Can we agree on that for today?”

“I object!‘ inserted Santino. ’Since ’XT’ is more frequent than ’TX’ in English texts, the entropy cannot be additive.”

“Not necessarily,” reacted Fred. He continued on the board.

→ Shannon extensive, page 131

“A factorizing joint probability leads to an additive Shannon-entropy,” concluded Fred.

“You are confusing me,‘ reacted Santino. ’So if XT and TX have very different probabilities, but EX and XE, further ET and TE about the same probabilities, then your factorizing hypothesis must fail. My ’examples’ on the short hand are: ’exercise’, ’execution’, ’expert’, ’exert’, ’excellent’, ’extrude’, ’export’, ’xenon’, ’xenophobia’, ’Xena’, ’xenobiology’.”

Santino showed his calculations on the board.

→ contradicting to extensivity, page 132

“The above contradiction – as I derived – shows, that the entropy of the English texts cannot be additive.”

Ego felt that it is high time to interrupt this *disputa academica*.

“Well, let us have a break. There are some refreshments in the room’s rear end,” closed Ego this session.

The party went to the tables. They ate and drank and talked. Ken tried his

50% level English text courting game on Ginny:

“TONITE E O OT TO EAT ITAIAN.”

“E A?”

“ON O AN I.”

“NO!”[2]

Tempering with the machine

Ken was not satisfied with this day. He could not sleep either. His thoughts circulated around the mystic machine in the cellar. Finally he stood up in the night and went to the antechamber with the disguised entrance to downstairs.

He found the touching spot rapidly, but it did not react. He felt anger. Suddenly a low voice behind his back alerted him:

“It works only with selected persons.”

It was Ms Kunz. She fixated Ken by a curious glance and then suddenly touched the spot. Then the entrance opened.

“I have to escort you,‘ told she, ’otherwise you cannot get back.”

“Thank you for your help!” echoed Ken. Suddenly one more shadow slid to them, Ginny arrived.

“I am also curious of that machine,‘ she declared, ’let us discover it together.”

So the small party of three approached the underground lab with the strange machine. Ms Kunz made light, Ginny sniffled around in the lab, Ken went directly to the machine. He opened the door and investigated the inside consol.

“Here, we have some strange script!”

All gathered to look at it. Ken presented:

DO NOT PUSH ME!

1. JGKB CQMGKVZ KB UXP BGXDKVH QYJZPVQJQKNZB

2. JGKB PQVWXC BYKWZB QPZ RXBBKSYZ SZJDZZV JGZC

3. JX HZJ SQMI JX JGZ BJQPBKVH XVZ KB BGZQP YFMI

“What is this? Some product code of the manufacturer?” asked Ginny.

[2] Tonight we go out to eat Italian. We all? Only you and I. No!

“For that is too long. I rather suspect a scrambled English text,” suggested Ken.

Meanwhile Ms Kunz had withdrawn herself in a silent English style, leaving the door to the lab open. She did not intend to interfere the decoding challenge for the two.

“Let us hope it is lossless coding, say letter by letter,” added Ginny, rubbing her forehead with the pointer finger.

“Even that would have $26! \approx 4 \cdot 10^{26}$ possible permutations,‘ exclaimed Ken, ’a hopeless case for random decoding.”

“We have to suppose that the coding was made by an intelligent entity,‘ tried Ginny to find the way towards the solution. ’It means we have to find out the clue, the very formula, which mapped the 26 letters onto themselves.”

“Yeah, let’s try to find it out. I make a shift of minus one, modulo 26, on the first 4-letter word: IFLA. Another shift with plus one: KHLC. It does no help.”

“Perhaps it is not a shift. At least not in the alphabet.”

“How else?”

“What about the frequency table?”

“Oh, let me try. Minus one shift in that table: JGKB $\rightarrow$ QWVP. The plus shift makes ZPXV. It makes no sense to me.”

“All right. Let us take a little distance from our immediate idea. The shift hypothesis is based on the old coding machines with rotating dials. As in the ENIGMA project.”

“What else can be a simple scrambling of letters other than shifts?”

“Hmm. Let me think now like a girl. What about mirrors?”

“Mirrors?!”

“I mean mirroring instead of shifting in the rotation of a dial. Take out, turn it and put it back in a retrograde manner.”

“Wow! Such a cruel joke! Who would be able to think this way?”

“I have some candidates on my mind. But let us try this method.”

“OK well, then in the English frequency table the most frequent E will be converted to the less frequent Z and vice verse. T to J and J to T.”

Ken did this work on JGKB making THIS. Then on all the rest. The following instructions occurred:

DO NOT PUSH ME!

1. THIS MACHINE IS FOR SHOWING ALTERNATIVES
2. ONLY RANDOM SLIDES ARE POSSIBLE BETWEEN THEM
3. TO GET BACK TO THE STARTING ONE IS SHEAR LUCK

"This at last makes some sense. Formally."

"The meaning of the message is another question."

"And the intention behind is yet another."

Ginny became somewhat provocative:

"Are you half as brave with unknown challenges as with changing girlfriends?" she asked unkindly.

"Women! Just seat beside me and watch!" reacted Ken to the provocation. And he pushed a big red button and pulled on the biggest arm to his right.

The doors of the machine got closed, lights went out and the two felt some dizziness for a while. They were taken by surprise.

The environment looked shifted. Ken had the strong feeling to be suddenly elsewhere. But it was just an unexplainable feeling. Ginny has moved slightly. *"Should this be an illusion, it is a good one!"* thought Ken. He tried to emerge from the seat of the machine.

He succeeded lightly. Too lightly, perhaps. Then he turned back to wake up Ginny. She opened her eyes and had a strange glance in them.

"Where are we?"

"Still in the machine. Let's get out."

The two stepped out into the lab, and *it* looked strange. Shabbier, dirtier, a used cellar room. Also the high-tech smell was missing. They moved like in a dream.

"Strange, isn't it? As if this would be another world..."

"Let us get out to the surface," Ginny suggested.

They went through the corridor, entered the antechamber. It was daylight, everything silent, covered by some layer of dust. As a postapocaliptic world.

"I do not like what I see,' whispered Ken. 'Can we now leave this place?"

"Let us go out and look for some vehicle."

Our adventurers did so. The terrace and the park also looked demolished. The stones were cracked and partially covered by moss, the lawns unmowed, being spotted with wild flowers and weed everywhere. An old pine tree just laid across the walking path, its stem already starting to rotten. Ken's car still parked at the terrace corner.

He walked to his car only to realize that this wreck is unusable. Its time has been gone for long.

"It looks we have to walk out."

Ken started to cross the park around the old castle, heading towards the former main road. Ginny hesitated for a short while, then she shrugged and followed the young man. They crossed the lawns and the woods, which must have seen better days earlier. Then a country road appeared. Ken contemplated which way to go, but Ginny continued her way to the right, so he followed. Everything was strangely quiet, no cars, no vans, no machine noise. For that matter no birds, no animal sounds, nothing. They marched through an abandoned land.

After a while they felt thirst and fatigue. They had to stop and get a rest. Then, beyond the next curve of the road they discovered a small hovel, offering shelter at bad weather. They sat down on the old and hard wooden seat for resting a bit.

"We cannot survive long without water and some food," said Ken.

Ginny agreed. She wanted to suggest to search for a well in the woods, when suddenly a bus appeared. It looked like a school-bus from old times, painted yellow once-upon-a time. The bus stopped, the door opened.

"May we go with you to the next village?' Ginny inquired. 'But we do not have money with us."

The driver, a middle aged man with some light brown skin and a three-days-beard answered with a faintly latin accent:

"This 's a comm'nity bass. None pays for. Get on guys, notime to talk."

Ken and Ginny got in and took a double seat on the right. It was a silent journey until the next stop. It occurred in a small town, in the center. An old town hall with two storeys in the right, in front of that an artificial pod, the water had an unidentified color between grey and black, spotted with dark green weed and algae. Everything looked used and malfunctioning. A bad dream.

On the left there was a small sitting place, with the large but dirty window reminding to a Mc-something for food and drink. They tried their best and entered. A tired looking girl stood behind a counter.

"What will be, guys?" she asked in a low voice.

“We would like two cokes and hamburgers, but we forgot our money home,” tried Ginny to order.

“Nun carries money any more. D’yu have your chips?”

“I am afraid not,‘ Ken was looking nervously around. ’Could you just give us two glasses of water? We became too thirsty...”

The girl shrugged and disappeared behind the back paravan. Soon she returned with two glasses of tepid water. They tasted like smoke.

“I’ve called the pol. Is something bad with you?”

Ken froze. “What? The police? Why?”

“Ya kno’, I had to. This is the law.”

In that moment and old and big police car stopped in front of the window. A solid looking sheriff emerged and came to the shop with determined steps.

“Hi, Sally,‘ he said, ’what matter?”

“These two have no chips. Strange.”

The sheriff turned towards Ken. He grasped his left hand and had an inspecting look at his wrist. He was astonished and rubbed his eyes.

“Where are you from? How is it possible not having the chips? In your age!”

Ken started to feel uneasy. Just when someone has the feeling that the dream starts to become a nightmare, but cannot do anything about it. Since it is just a dream, just a dream.

The sheriff begun to talk in a didactic way, as if quoting some well learned text.

“In our society we all are equals. Every person has the same rights, owns the same common property for all and there are no differences in income or wealth. But, to avoid chaos, every person has to be traced at any time. We are responsible for our citizens.”

Yes, the nightmare started. Ken tried to awake from this. He concentrated his thoughts to his wish to wake up. Stronger and stronger. He closed his eyes. Gradually the lights went out, the voices disappeared and a cool air touched his forehead. Can it work?

Meanwhile Ms Kunz fetched Ego. They saw Ken and Ginny sitting in the device’s seat, and a strange, hardly visible bundle of extremely thin silver axons connecting to the back of their heads. Ego went closer and with a determined move he interrupted these nanocables. Ken and Ginny opened eyes and looked like awakening from a dream. Somewhat disoriented, Ken looked in Ego’s face.

“Oh, what was this! A dream or a nightmare?”

“You seem to have tempered with the machine. That you shall not do!”

“Sorry, Ego. But I am an engineer so I have to touch machines, consoles, controlling keyboards and such. Let us talk about it tomorrow.”

“It is already tomorrow. Take a shower and join the common breakfast. This is an order.”

Ken made a face to this claim, but he obeyed. Ginny followed without a word.

Discussion at breakfast

The breakfast was the same as always, good enough. Unlimited refill of cafe Americano, a brownish fluid with a big amount of hot water, milk and other dairy products, fresh bakery goods, eggs, ham, peppers, tomatoes – already a collection of plants and animal products from the same continent. A wise move with respect to sparing unnecessary burden for the environment.

The scientists and Ego with Ms Kunz ate together at a big table. Ken was reporting about his night experience.

“I think I did it. I was sitting in the machine, Ginny beside me, and I switched something on. Than we had an adventure, which looked extremely real to me.”

“It must have been your dream,‘ exclaimed Ginny. ’I was sleeping in my bed the whole night. No way to sit beside you.”

“But Ms Kunz brought us there. I cannot open the secret door myself,” continued Ken and was gazing at Ms Kunz.

“I can assure you Dr Kenson, that we did not have any common experience in the night,” was the reserved answer from the other side of the table.

Ken felt insecure. *Could it have been just a dream?* He shrugged.

“I cannot be sure whether it was a dream. Since both of you are denying... Nevertheless, I remember being in the outside country. And it was strangely different...”

“Perhaps we can discuss the essence of your dream, possibly a message from your subconscious,‘ suggested Ego, ’in terms of our research goals at the morning meeting.”

When they started their meeting the scientist were all interested in Ken’s story. He stood up and talked.

“It was a strong experience on my side! I dreamed – let me suppose this in the absence of any contradicting proof – that I went to the cellar and sat in the machine. Ginny was with me in that dream."

"My dreams are different..." noted Ginny.

"In any case, we decoded some message claiming that the machine is an alternative reality simulator. Then I pushed and pulled some levers and everything had changed."

"What everything?" inserted Santino.

"The ambient light, the lab, the machine itself. Furthermore, we went outside, and the whole world looked different."

"Better?" asked Ludvick with a dreamer's gaze in his eyes.

"Not better. Rather worse. As a dystopia in hundred years from now."

"Whose dystopia?" coquetted Ginny.

"A shabby world. Dirt and ruin, nothing looked new, everything used over and over again, until it collapsed,' continued Ken. 'We..., I visited a small town, and got into trouble."

"Because of women?" inquired Vilmano.

"Because of having no money. No, not quite. Because of not having a chip built into my wrist."

"Under your skin?" wanted Ego to know.

"I do not know. Everyone gets such a chip after birth and uses it for identification. No money is used, otherwise."

"That sounds practical. Your debts and earnings are automatically recorded. Then there would be no need for physical money," Vilmano agreed.

"This is not all. I was arrested by the local police, and the sheriff lectured me."

"That could serve you..." noted Rudolph with some malice.

"I was lectured that absolute equality is the basic social principle. That everyone has the same income or wealth. Everyone is in the same state, the inequality is factually zero."

"This is communism at its textbook appearance," noted Fred.

Santino raised his hand, "If a distribution is singular, being nonzero only at a single state, then the entropy, I say all entropies, are minimal. And this minimum better be zero."

He jumped up and replaced Ken at the whiteboard. He wrote formulas demonstrating what he had just said.

→ zero entropy PDF, page 132

"I still do not see why should it be zero," shaked Ego his head. Some others gazed also without the spark of understanding.

Santino continued by showing this for some common formulas.

→ ranking of entropies, page 133

Vilmano arose. He replaced Santino at the board.

"It is fascinating, that the measure of inequality in economy, the Gini index, is also zero for the singularly communistic wealth distribution."

"Could you show us the formal background of it? We need some introduction into the Gini index. What kind of abbreviation is this GINI?" interrupted Ego.

"Oh, the Gini index is named after Corrado Gini, an economist in the beginning 20th century. Nevertheless, as a mnemonic, you may view this as a mosaic word, too: Gross Inequality Natural Index."

Vilmano then took the black pen and wrote some simple formulas on the board.

"The inequality is defined by measuring the average size of differences between randomly picked values."

→ Gini index definition, page 133

"So if no differences are allowed, the Gini index is exactly zero,‘ concluded Vilmano. ’In this respect it is akin to the concept of entropy, however I do not see any common root for these two distinct ideas."

"Please, do not forget that the concept of entropy is more general than the Boltzmann formula. We can have other expressions, just very general constraints on its behaviour must be satisfied," added Santino.

"Can it be a program for our research? I mean to explore possible connections or analogies between the Gini index and entropy?" put Ego the guiding question into the air.

All showed some excitement. To find something, for we have no obvious reason, but also cannot immediately refute, is the very challenge for theorists.

"We might generalize both formulas until we find a common form," suggested

Fred.

“We can check the Khinchin axioms, designed for the general entropy, on the Gini index,” contemplated Santino half loud.

“Or we may evaluate income data sets both with Gini index and entropy. And then we shall see whether they behave at least agreeable, synergic, even if not equally,” tried Ginny to put another aspect into the discussion.

“Surely, can we collect data on individual incomes from the net. The question is how reliable those are. How many fallacies, distortions, hidings are built in publicly available data,” cooled Ken somewhat Ginny’s suggestion.

Finally Ego raised his hand. He waited a few seconds to achieve attention. In the sudden silence he announced:

“Very well, next we review the generalization efforts for the Boltzmann entropy, and investigate – if there are some – the entropic properties of the Gini index. But now we need a break.”

Just when all rose, Ms Kunz came to Ego and told him something privately with low voice. Her face was pale.

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Chapter 4

Alternate Formulas, Alternate Realities

A dialogue with Inspector Holmes

Ego and Ms Kunz hastened out of the dining room. At the entrance Inspector Holmes was waiting. The three disappeared behind a middle sized door beside the staircase, leading to a small library.

Ego offered a seat for the Inspector on the leather sofa, himself took another armchair nearby. Ms Kunz soundlessly brought a tray with two glasses and a bottle of whiskey. Then she looked at the Inspector and nodded. She fetched a soda maker, too.

The men drank and kept silence for some seconds. Then they started together:

“This is an awful news, Inspector...” “I do not bring good news, Sir...”

“Please, let me be informed. You first.”

The Inspector took a deep breath and told about the strange thing that happened. The man, they had found at the machine in the cellar and brought into a hospital two nights ago, has disappeared.

“What do you mean by ’disappeared’? Escaped from the guarded room?”

“Yeah. And without any trace of crossing the door, the closed windows, or breaking the wall...”

“You mean, as if he had used the fourth space dimension?”

“I do not know what is that. Is that some hidden tunnel?”

"Well, in a sense... it can be a tunnel. Or something else, we have no idea of."

"Fact is that the person was not found this morning in his room, nor elsewhere in the hospital building. And nothing left behind. Not a molecule we can trace."

"It is strange. Such big objects, like a human body, usually do not disappear."

"Why, small things do?"

"Very small things, like elementary particles, which are energy quanta of fluctuating quantum fields, may jump in space and time. They do not propagate smoothly, only that is the most probable path. As far as we understand."

"But should not the impossible be the 'least probable'? Or even of probability zero?"

"This is more like philosophy. Impossible is even a stronger statement than probability zero. Quantum tunnelling for everyday sized objects must have a very-very low probability. Extremely low. On the average much less than one such event should have occurred during the whole lifetime of our universe since the big bang."

"Good. Than that man is a clever magician and tricked us."

"Can he still be in the room? You just did not see him? Hiding in a clever way?"

"Oh, now I can choose between invisible and impossible."

"Sorry, Inspector, but solving mysteries may rather be your job, than mine. However, this case reminds me of something."

"What then?"

"That the man also appeared at the machine in the cellar lab without previously being there. As if he'd just made a short excursion in our universe... from somewhere else."

"From somewhere else?! I am afraid my legal competence ends in this universe. Sadly."

Then Inspector Holmes rose and walked out from the library. He took his coat and hat and left the castle to join his attendant in the police car. They gave gas and left the estate immediately.

Ego was left with a strange feeling. *Was this mysterious person one of them, erring back and forth between alternate universes?*

Alternative histories

When Ego entered, all the seven scientists were already in the discussion room. He marched to his place with steps somewhat heavier than usual. Then he opened the discussion.

“Dear Doctors and Professors! Now we need your keen minds.‘ He kept a brief pause to enhance the effect. ’Is it possible to have alternate histories of the whole universe, or copies of the universe, or different universes?”

“Are you aiming at the ’multiverse’ theory?” asked Santino.

“Theoretically each element of *’the’* history might have happened in another way. Since we do not know any constraint which would have determined that it had to happen exactly the way it happened...‘ inserted Fred, ’any theory assuming alternatives is a logical step.”

“Logical, leading to where?” picked on him Ludvick.

“Leading to a better understanding of why exactly that path was taken, which was taken,‘ continued Fred, undisturbed. ’It is just so in case of the human history that the plethora of phenomena and their alternatives cannot be nailed down due to our very rare knowledge about background data behind the few recorded events.”

“What you are saying is the physicists’ way of thinking,‘ added Ken. ’We also need to know how to make it better, how to manipulate our path for a better future while knowing all the constraints. If it is possible.”

Ginny changed her gaze from one man to another. She gently shook her head to let her dark reddish-brown hair waving. She hooked into the dispute:

“Again your abstract theories! Why do not we study first some examples. Some examples of alternative pasts of our presently known history.”

The group quieted for a while. Than Santino took the first challenge.

“The easiest is to establish alternatives to mathematical models. Some formulas can be generalized to a class of more wealthy expressions and only a special case seems to occur in nature. In most of our formulas we have parameters. Changing them changes the result. In our variational principles we have optimization criteria: for the action in dynamics or for the entropy in statistics. That classically distinguishes the best alternative.”

He held for a moment. Then continued.

“We want examples first, not theorems. Quantum physics embraces the idea of alternatives. The solutions of our classical equations are only the most probable alternatives, but not the only ones. It emerges only in the limit of a very large number of possibilities that *practically* only the solutions happen. That is classical motion."

"The mechanism behind this 'wonder' is also worth of a discussion. In quantum physics the alternative paths of a particle interfere with each other. The 'rotating phase' factor, $\exp(iS/\hbar)$, describes the enhancing and cancelling mechanism due to interference. And the winner is ... exactly those for which the action, S, changes the least, with other words it is stationary. In essence the solution of our equations makes the action stationary."

"But why would we accept that everything behaves like this, out of the blue constructed, quantum physics?" wanted Ego to know.

"We do not. It is just a suggestion for a working hypothesis..."

"You know, what Newton said about hypothesis..."[1] added Ludvick, amused.

"Dear Colleagues, please let us get back to a serious discussion,' suggested Rudolph. 'I propose to discuss first examples for alternatives in human history; what could have happened differently and whether it would had led to a very different present world. Then we may discuss alternative formulas and their consequences for our world-view in physics."

"And then we may discover alternative science in the alternative world,' added Ludvick. 'My favourite will be a dummy science without the concept of atoms – as many believed until the end of the 19th century."

"Well, let me start with such a *Gedankenexperiment*," proposed Rudolph.

Rudolph's history

"My human history starts to deviate from the one we know in 1941, in Nazi Germany, before the attack on the Soviet Union. There was yet another attempt of assault against the *Führer*, which did not succeed, and was not recorded in our history. By chance Hitler was not poisoned as planned, due to his vegetarian diet. But he became weak for days and got into a hospital."

"Why do not we follow an alternate history when one of the assassination attempts was successful?' asked Vilmano. 'That would have definitely changed everything!"

"We want a small perturbation. That is what checks stability, not a big disruption from the beginning," reacted Rudolph. Then he continued with his story:

"In the hospital, of course, he received the best treatment by the best doctor.

[1] "Hypotheses non fingo" – I do not deal with assumptions.

As you may guess the best doctors in Germany were Jews. This guy, let me call him Dr. Roth, had a research project beside his everyday work in the hospital. He injected a cocktail to the *Führer* which contained not only pain reducing and wound-healing components, but also something which cured his paranoid personality. Just slowly, molecule by molecule, nerve cell by nerve cell, connections by connections in the brain area where the emotional base of the personality resides, changes took place."

Rudolph held for a second to drink a bit. Then went on.

"Curing paranoia and schizophrenia was an unwanted, random effect. Yet, it worked. Hitler became 'normalized', after a few weeks he started to feel and think and behave more cautious. He used his intelligence for reaching balance between expansionists plans and the deliberation of possible enemy forces. Most prominently, he did not attack the Soviet Union. He yielded to the Ribbentrop – Molotov pact."

"This already has changed the history, first that of WW2. The Germans concentrated more force on the west front, uniting continental Europe under their yoke – more effectively and more permanently as Napoleon did. Hitler had made a peace treaty with the English, before the US could have got involved in Europe."

"Now, if you think these were big changes, this cannot be all. In such a scenario Hitler also stopped the persecution of Jewish scientists, turning them into figures of a national research project on nuclear energy, led by Heisenberg. Einstein, already in Princeton in the US, did not write his letter to Roosevelt, since Leo Szilárd did not go to him. He returned from the US via London to Berlin after the European peace, say in 1943. So the Manhattan project in Los Alamos never happened."

"Instead, Soviet Russia and the Germanized third 'Mitteleuropisches Reich' started to pursue common research on the civil use of nuclear energy and rocket development. Hitler was asked by the Americans to mediate towards the Japanese, then events at Pearl Harbor could not be undone in this scenario. However, they, the Americans, did not have the atomic bomb as an argument on their side in this universe."

"Finally Europe should have promised a big amount of payment and modernization to Japan in change for their peace with the USA. Japan kept Mandschuko, an occupied territory in mainland China, and numerous islands in the Pacific. The world became pacified, but full with armed military everywhere."

"Now. We wanted to correct history, starting with a mild change. But it is not excluded that we made it worse. In any case the deviations grew and grew. Nothing like a stable history occurs in this example."

Fred's history

Fred raised his hand.

"Let me go through another example, when the small perturbations happen closer to our day. Then the changes might also be more delicate."

He paused for a second to gather his ideas. Then presented the followings:

"My tiny deviation happens in 1986. Gorbachev and Reagan met in Iceland, Reykjavik, for a summit about the fate of the world. Gorbachev wanted to finish the cold war by a 'double zero' solution of strategic nuclear disarmament, in order to make him able to slowly meliorate the living standard of his countrymen while keeping the Soviet Union. Reagan wanted to end the cold war by crashing the soviets to extreme poverty, when they give up contesting against America in armament. His leverage was the Strategic Defense Initiative, SDI, a research project about extending the US's military rule to space around Earth. Gorbachev wanted to stop this, the Russians and their forced allies not having enough resources to keep in line with this new development."

"The summit ended without an agreement in our universe. Nevertheless it already signalized the path leading to the dissolution of the Soviet system and a major rearrangement of the world's power balance. Gorbachev was the one and only leader who restrained from a military invasion in the Eastern Block when the European countries, one after another, started to turn their back to Marxism and the social experiment associated with it. He also agreed to the reunification of Germany, a country in the heart of Europe, divided since WW2. As a result he became extremely popular in both Germanies, and increasingly unpopular in Russia. And that led to his resigning in 1991 and to the disintegration of the Soviet Union, to the end of a global player until then."

"So what tiny change might have influenced this path in our history? According to my assumption Gorbachev's mood on the first dawn of this negotiations was changed by a tiny sun flare. It went unrealized, of course. But then his wife was unexpectedly caring for him on that morning and warned Mikhail about some persons back home, in Moscow. Mikhail knew that he cannot be too soft with the Americans, but at the same time his focus on stopping the SDI had shifted to a less significant position. He was thinking more on who would want to replace him back in Russia."

"As a consequence on the second day he agreed not to stop the research program about laser guns and similar sci-fi weapons in the US, if the results of these early experiments were communicated openly. Certainly, based on past experience, he felt that the US cannot be trusted on this, yet he evaluated this as a payable price for more peace and opportunity for his own land and people, whom he was responsible for."

"In this way Gorbachev had achieved a better position in the Reykjavik treaty in this parallel history. And he used this advantage wisely. Indeed, the soviets gained more time to harvest the fruits of their perestroika, their reconstruction of the socio-economical system. They moved peacefully from Marxism to social market economy and succeeded in installing reforms without pauperizing the middle class. Economic crises were much less deep in the 1980-s and 1990-s."

"As a further deviation from our history, the Eastern Block also did not dissolve. They gained some political independence from Moscow, but were more and more integrated in the emergent Euro-Asian market. The latter was based on a new Russian-Chinese agreement, gaining access to the largest consumer market of the world. In 1991 Boris Yeltsin did not challenge Mikhail Gorbachev. Gorbachev was followed by Vladimir Putin after the 2002 presidential election in Russia. They transformed the Soviet Union peacefully into a Union of Social and Democratic States, to which East-European countries voluntarily joined during the period 2012-2036. Germany, however, preserved his political independence, although became heavily involved in the profitable exploration of Siberian gas and oil."

"That world you would not recognize any more. Even official borders, names of countries and regions, the name of money were different. The Czechs would have approached the Franco-German European Union, which included France, the Benelux states, Scandinavia, Germany-Austria, Northern Italy and Slovenia. Poland, Slovakia, Hungary, Croatia, Serbia and Romania build a V6 collaboration inside the Extended Euro-Asian Alliance, trading both to West and East, and sharing a constitutional culture with the emerging Euro-Asian superpower."

"Please, stop!' interrupted Ginny. 'You went far on another path than the reality, but all your steps were random and unestablished ones. Infinitely many different scenarios may happen and all diverge from each other. Classical chaos."

"You wanted experimental examples instead of theory," warned Fred.

"But we need a good example, which teaches us where to look for a theory, " replied Ginny.

"Perhaps we should agree on some qualifiers, some measurables we could follow,' inserted Vilmano. 'Stories, as the name history suggests, are subject to interpretation. It is conjecture, without science. Worthless narrative."

"Perhaps we should restrict ourselves to the history of physics. Possibly an easier task to complete..." suggested Santino.

Ginny's history

"Before we do that,' said Ginny, 'let me present a counter-example. A quite different history for about 100 years with a result similar to what we have right now."

“An interesting suggestion!‘ exclaimed Ego. “A kind of elasticity of the fabric of history.”

“Let me concentrate to European history, around 1910,‘ Ginny continued. ’Germany and Austria-Hungary, the core central powers, were not destined to loose WW1, even if the war was not to stop at that date, already.”

“What could have happened differently?” asked Vilmano.

“For example that Italy had not change sides in the midst of the war,” ironized Rudolph.

“I plan to consider more subtle changes,‘ corrected Ginny. ’Italy had its conflicting interests, as a background fact. Let me tell my story, first.”

“Austria-Hungary had a weakness rooted in the multicultural, multinational and ethnical composition of this landlocked empire. Most prominently they would have needed a more balanced representation of the ’nations’ inside the Habsburg empire.”

“In fact, Franz Ferdinand, the heir to the throne of Franz Joseph, was a promoter of the idea of Austro-Croatian-Hungarian trialism, ACHT,‘ added Fred. ’But the old emperor resisted, permanently.”

Ginny took the speech again.

“Indeed this is one of the things which could have happened differently. In four years the old emperor father and the resistance of Hungarian nobility in their parliament could have been overcome, with patient and convincing, first of all economically convincing arguments. The ACHT could have been stabilized from inside, before standing for the external challenge.”

“Nevertheless, some things never change. Serb nationalists would have assaulted Franz Ferdinand as well have in our history. This attempt, however, might have not been lethal. The trialism also strengthened Croatia, modernized its army and the infrastructure.”

“Germany, on the other side, could have pushed the scientific and technological developments even stronger. The Kaiser-Wilhelm institute chain was a great invention. They introduced in this scenario the mobile artillery earlier, and invented, say, a chemical weapon not killing people, but destroying armanent. The latter better be a fluid than gas, so the capricious wind could not reverse its effect back to the launcher side.”

“A stronger central power alliance and a weaker *entente cordial* would have been resulted in a different outcome of the hostilities. Though Italy, Serbia and Turkey would have played their roles similar to our history, the communist revolution would have switched Russia off from the war activities between 1917 and 1922. And here we may speculate about further changes: Germany recognised the thread of communism more firmly, and offered a helping hand to the 'whites' in Russia by stopping the eastern front battles."

"I want to make it short: if WW1 ended with the victory of the central powers, with a weak France and England, an economically ruined Italy, a modernizing Turkey and Russia and a stable and flourishing Germany, Austria, Hungary, Croatia, allied by Bulgaria, Moldovia, Lithuania, and other Baltic areas annexed by Germany, together with a freshly independent Poland at its historically more eastward territory than in our universe, then what would we have?"

"A German led strong European Union utilizing the strength of its Eastern components?" asked Fred.

"Exactly. And even the borders would not look much differently from the today's world. Nationalism would not be aggressively suppressed until explosions occur, but channelled into a system of both balance and sustainable development; the communism would not be as popular as in our history, since the social justice would be based more on political and legal equality guarantees than on forced redistribution."

"Indeed, this is what we may call elasticity,' summarized Ginny, 'various event chains leading to a similar result. Alike quantum paths tending to the classical solution in a dynamic world, given enough explore-able alternatives. Should there be laws, how else can they act as statistically?"

"Well, then,' emerged Santino to speak, 'but what is the governing principle behind forming the statistics? Shall we crown entropy as the god-principle behind the laws of nature, and, by being there, of naturally emerged complex systems? Can it quantify inequality for good or bad?"

Generalized entropy

"We have axioms selecting out the Boltzmann form for the entropy – probability relation,' intervened Ludvick. 'It will be hard to violate them."

"Indeed,' replied Santino, 'if we generalize the formula we also may generalize one or other axioms. We may release for example the requirement of extensivity."

"In fact the Rényi entropy satisfies the additivity requirement for factorizing probability,' added Fred his remark. 'It was constructed to behave so."

"On the other hand it is not an expectation value in general, in contrast to the Boltzmann entropy,' took over Santino again. 'The Boltzmann entropy is the special case for $q = 1$ Rényi entropy, but for other values of $q \ldots$ it is a nonlinear function of an expectation value."

"But what is the physical meaning of this mysterious q? How mother nature adjusts its value?" wanted Ginny to know on the spot.

"Let me try to answer this question by showing a comparison of the canonical probability distributions in both cases. I start with Boltzmann's idea of comparing volumes in the abstract phase space and obtaining the probability to find a part of a system in a given state by this. Physicists view this as blind chance."

→ q related to variation and finite size, page 134

"Summarizing, the deviation from the classical Boltzmann entropy formula in this special parameter q measures finite heat bath capacity and fluctuation effects at the same time,' abstracted Santino the results of his just presented calculations. 'The conventional view estimates the fluctuations by declaring $q = 1$, equivalently it declares that the typical relative variance in the inverse absolute temperature is exactly $\Delta\beta/\beta = 1/\sqrt{C}$, inversely proportional to the square root of the size. But it must be a question of observed facts, whether this is true or not."

"Wonderful!' exclaimed Fred. 'Then math was just waiting for finding natural phenomena which fits the human logic. Some of them do the favour of exact compensation, some others not?"

"Indeed, the $q = 1$ or not is a case by case question at this level. No law of nature seems to hide behind it."

Now Ego interrupted the science discussion.

"Professors, we are by now far enough from the original research goal about inequality. I declare this session to be finished."

By that was told, most of the people rose and went out to straighten their limbs. Also lunch and the after lunch leisure time followed. Being a sunny day, many took the opportunity for a walk in the park, or for a short sunbath on the terrace with a cocktail in the hand. Ludvick and Ginny changed quickly dress and jogged around the castle.

Ludvick could think the best by jogging. Fresh, cool air, sunshine, the rhythm of the run and the monotony of turns around the huge building brought his brain in a semi-hallucinating stage. He performed conversation with self. *A non-extensive entropy follows a non-additive rule by composition. The next generalization of addition, and for that matter of multiplication, too, is group theory. Existence of a trivial member of the group, the "unity", invertibility and associativity of the group operation, rendering a new group element to any pair inside the group, are the defining properties. Invertibility, searching back an original element from the result of the operation and from the other element, could be abandoned as a requirement. Then we dealt with a semi-group.*

Parallel to these thoughts he realized Ginny running ahead. Ludvick was slower, since he liked the epic thinking during his sport. He realized the curves of the woman's shape, yes, indeed, the hip to waist ratio... is different. Ludvick felt that he likes this, he finds the contours of the running woman harmonic and pleasant. Why is that? One of the secrets of the human evolution. *Could his instinctly felt pleasure on harmonic forms be somehow related to the statistics, that more men than women are attracted to theory? Well, the evidence so far is insufficient.*

Ludvick decided to show his formulas on associativity in the afternoon session. Who knows, perhaps Ginny will pay more attention on him afterwards...

In an hour all gathered in the lecture room. Ludvick raised hand and suggested to show a few formulas. He wrote on the white table.

→ associativity and formal logarithm, page 135

"Hence any group entropy, where due to associativity the grouping step order would not matter by the repeated composition of bigger and bigger systems, can be described by an additive quantity, the formal logarithm of the group. And then one may call *this* additive construct the 'real' entropy," concluded Ludvick with a victorious smile.

"Congratulations, Ludvick!' reacted Santino and went himself to the whiteboard. 'You may take your seat, now I'll take it over."

Ludvick was disturbed. Santino complimented him, but also dismissed him as a young, inexperienced student. He looked secretly at the audience. Ginny seemed not to be there, rather dreaming of a parallel universe herself. Others looked at the table, some nodded, some yawned discretely, with closed mouth but tickling muscles in their faces. Ludvick gave it up and returned to his seat. Look at the master...

"So even if the entropy would not be additive, we can construct a formal logarithm function of it, which is additive. Fine. That also means that in the generalized case we use $K(S)$ instead of S, and declare that the measure of unbalanced, non-standard fluctuations will lead to the $q_K = 1$ based on $K(S)$. Let me show the details," continued Santino with his Latin momentum.

→ $K(S)$ based non-additivity parameter, 150

"Whoa! Not the simplest formula of all..." noted Ginny.

"Let us discuss first some particular cases,' replied Santino. 'In the large environment limit, underlying the classical thermodynamics too, we may assume $1/C = 0$, i.e. an infinite heat capacity for the heat container. Admittedly an idealized case."

"I am afraid then everything will be rendered to trivial," tried Ludvick to reverse

Santino's earlier behaviour towards him.

"Not exactly,' told Santino and then turned to the board. 'We keep the fluctuations, though."

→ Fluctuation dominance, page 137

Finishing and encircling his entropy composition formula, Santino turned to the audience to explain what was achieved.

"*Summa summarum*, in the case when the environment is ideal, when it acts as if it were infinite, but the variance in the inverse temperature due to fluctuations among different realizations of the system are still appreciable...' Santino took a breath here, '... then we arrive at the Tsallis–Abe non-additive composition law for the entropy. Naturally our generalized, better told modified entropy, $K(S)$, is additive."

"Can we get back the classical results in some limit?" asked Rudolph, showing a genuine interest.

"Indeed. For $q = 1$ one chooses $q_K = 1$ and then $\alpha = 0$. That recovers $K(S) = S$ and the additive composition law. In general, any choice of $q_K = q$ does the same, no reason to deform the entropy if someone is consent with the original value of the q parameter."

"What about the other extreme?' wanted Fred to know. 'When the variance of the temperature is overwhelming, in spite having huge and ideal reservoirs."

"Then we differ from the classical formula. By the way, this extreme limit is also parameterless: for $\alpha = 1$ we simply have $K(S) = \ln(1 + S)$. It is a complex, hierarchic limit: $\Delta\beta^2/\beta^2 \gg 1 \gg 1/C$. The composition law belongs to $q = 1 + \alpha = 2$."

"Cannot we solve the general case, as well? Finite environment, finite variation in the temperature and deforming a $q \neq 1$ value to $q_k = 1$?" Fred looked as having some trouble, he concentrated heavily.

"Surely we can. But in order to understand its full potential, let me first demonstrate another consequence of using the deformed entropy," proposed Santino. And he showed another couple of formulas on the whiteboard.

→ deformed entropy – probability formula, page 137

"So the deformed logarithm formula is the key not only to the composition rule, but also to expressing the entropy – probability relation for non-additive systems," concluded Santino his capricious derivation.

"But this formula is not the Tsallis or Rényi result!" exclaimed Fred. This statement caused some uproar in the audience. Santino put down his pen and straighten himself a bit.

“No. That formula we gain back in this route in the opposite extreme, when the finite heat container effect dominates and the superstatistical fluctuations are diminishing,” explained patiently, with a somewhat tired voice.

“Certainly, there can be different superstatistical fluctuations, not stemming from the expansion of the exponential of the deformed entropy. If they are god given, meaning at the scientist’s will, then any result can be reconstructed. However, what I am trying to present here is a consequent effect, where both the deformation of the additive rule and the entropy formula stems from the same and only derivation.”

“You could convince us easier, if we would see the general derivation,” suggested Rudolph in a deep and friendly voice to make peace among the cocking men. Some laughed. Santino turned and presented the general derivation.

→ General deformed entropy with constant parameters, page 138

“This formula is quite involved,‘ noted Giny, ’but does it lead to any known one in some example?”

“Indeed, this formula reproduces the Tsallis entropy formula in the small superstatistics limit.”

“Could you show it in a few lines?” inquired Ginny.

“Com’n, even I can do that!” declared Ken. People laughed again. Ginny’s face turned red slightly, and she had thrown a disapproving look onto Ken.

“Then show it, you engineer!”

Ken changed Santino at the whiteboard and he cleared it with cautious, slow motion. Then he smiled and wrote the missing piece of the full derivation.

→ the missing piece, page 139

Some looked excited, some exhausted. Ego frowned.

“Professors, Professors... and Doctors, and,‘ Ego looked at Ken, ’Engineers. We did a long way to get a distance form the classical theory of entropy. But I do not see how did we get closer to the research goal of understanding and handling inequality.”

Most researchers shrugged. Some made poker face, Ginny shook her hair. Finally Vilmano tried to save the situation.

“Well, fundamental inequality statements argue with the entropy. Therefore, it is important to know how solid is the logarithmic formula we knew, or what properties a generalized formula will keep."

"Furthermore, there are some formulas which are better than others,‘ added Fred. 'The Rényi formula is additive, the Tsallis formula is an expectation value – average in common language."

"At the end we shall return to the discussion of the Gini index,‘ suggested Vilmano. 'And I am waiting for an analogous discussion, revealing several ways to establish its properties. Is there a second law of thermodynamics governing the income inequalities? Or any other principle? Other than human malice?"

"Very well, tomorrow we change the discussion topic somewhat. Otherwise we all seem to be ready for a dinner leisure," closed Ego this session.

Chapter 5

Imagine, No Atoms

Ludvick's turn

Ludvick was upset. Not only that Santino handled him as a child, also Ginny showed no warmth towards him. Ludvick had a sensitive soul. A very sensitive one. He had the habit to brood over real or just imaginary malice towards him. It was easy to offend him, although he did not show.

And now his ego was wounded. And that ached.

This affair ruined his evening: he was absent minded over the dinner in the common dining room, he sank in his thoughts deep. At the same time, he had an antenna to evaluate his own offendedness, as if looking from outside. *Am I ridiculous? Imagining more than it happened? Shall I get it over?* The question was how.

He could not sing with Vilmano and some others, he did not drink at the bar, he retired early. He also could not sleep in his bed. Instead, he managed to get into a hallucinogenic stage, losing contact with the reality, more than a daydreamer, moving on a thin ice of loosing consciousness. No sharp border can be realized when one falls asleep with an overly tired brain.

A damped knock on his door woke him up – at least this was what he felt. Without making light he opened the entry. An elegant perfume overwhelmed him.

"Ludvick! I want to talk to you," whispered Ms Kunz, or someone with Ms Kunz's voice.

"Uhm, please..." Ludvick was caught by surprise.

"Not here. Please, come."

Ludvick was guided by a cosy, warm shadow in dark corridors. He was disoriented, but it seemed they sank deeper, perhaps into the basement level. Yes, after a short while they ended at the mysterious laboratory.

“Here we will not be disturbed,” suggested Ms Kunz and made a dim light. She seated herself in the machine and invited Ludvick to follow. Ludvick seated close to her, fully being aware of the presence of the woman’s mature body.

“So what is it about, Ms Kunz?‘ started Ludvick a conversation, partly also to govern his own thoughts back to the cold and secure universe of abstract intellects. ’Can I help or explain something to you?”

“You need more phantasy, Ludvick,‘ answered Ms Kunz. ’Call me Emma.”

“So what is it you want to talk about, Emma?” tried Ludvick to stay in the field of talking. He had bad feelings about the possible consequences of an affair here and now, and on the top of that with the secretary of the boss. Especially when the boss was Ego Zeus.

Ms Kunz, Emma, sighed. She whispered with lulling voice, “You see, Ludvick, you are offended on very minor impoliteness by Santino, you seem to aspire after the attention of Ginny, without sufficient hope to be noticed, and at the same time you do not mind to offend me...”

“Oh. I did not mean...”

The protest of Ludvick was weak, very weak. Ms Kunz moved, and she – perhaps inadvertently – switched on the machine. The connection with the background AI was initiated and Ludvick felt an increasing dizziness. He felt falling into a dark and huge space, and lost the conscious feeling of having the company of Emma. If it was a dream, it abruptly changed.

Suddenly brightness flooded the room. Ludvick squinted. There were some people around, in white robes, they looked like scientists around 1912 in a Kaiser-Wilhelm-Institute. He stepped out from the machine, with a weak head, somewhat uncertain. He wanted to make a step forward, but collapsed.

“Wer sind Sie?”[1] Ludvick became conscious again. The questioner was a woman around thirty. Certainly, Ludvick understood German, but he still assumed being in America. So he answered in English.

“My name is Dr Holzman, Ludvick.”

“Are you from England?”

“No, from Austria. *Ick bin aen Weenair.*”[2]

“Why do you speak than English to me?”

[1]Who are you?
[2]"I am from Vienna" – with a slight dialect.

"Sorry, I thought... I'd be in the US."

"Certainly you are. *In den Vereinigten Staaten von Euro-Amerika.*[3] And here German is the official tongue."

Ludvick felt alerted. *What? Another parallel history?* It seemed so. Perhaps not only a history, but a whole universe parallel. With different laws, therefore.

"For your sake we can talk in English. I have some relatives in Aberdeen, and remember the language quite nicely. Though I was young then," offered the young woman in the white robe.

"Thanks, let us do it. But you are still young," tried Ludvick some flattering courtship.

"You have appeared so suddenly in this experimental machine. We shall have to report on it."

"I have no objection. I am afraid it would not count if I had any."

"Correct. Can you stand up and walk now?"

"I think so." Ludvick stood up and after a deep breath his limbs started to obey. He walked.

"Fine. We go to Professor Grünstein. He is the *Chéf* here."

They went through the corridor, emerging to the antechamber. The castle looked somewhat different, with gas lamps on the wall, more massive furniture around. The style was rather solid than fine. The professor's office was at the place of the small library – as far as Ludvick could realize. When they entered Ludvick's escort started in English to subtly inform the professor that the visitor – on unknown reasons – would prefer this language.

"Herr Professor, this young man appeared in the machine downstairs. He told his name were Holzman, Ludvick Holzman."

The professor looked at Ludvick with a genuine interest. He was examined like an exotic specimen collected by a tropical expedition, like some freak in an asylum for insane.

"But, as far as I was informed, we did not start the experiment yet!"

"Correct, Herr Professor. He just appeared. It was not *our* experiment."

"Well, we have to investigate this case. Do not tell others. I order an information blockade. *Verstanden?*" [4]

[3] In the United States of Euro-America.

[4] Got it?

“Yes, Professor Grünstein. I tell this to the *Personelle* at the Laboratorium, immediately.” The young scientist turned and left the study room. She closed the door resolutely, with somewhat more force than it would have been barely necessary.

Ludvick felt his chance to influence the events. He addressed the professor:

“Herr Professor, können wir privat sprechen?” Sir, may we talk privately? For the sake of exclusiveness – in English?

“Hm. Rank has its privileges. You are a talkative stranger. Can *you* explain your appearance in my most secret laboratory?”

“Not entirely. But I have a theory about it.”

“What impertinence! It is *me*, who should have theories here.”

“I beg your gracious pardon! But since I came from the ’other side’ of the machine, you may find my view interesting.”

“Go on,” agreed Herr Grünstein indulgently.

“So my theory is as follows. This machine, we do not know why, opens tunnels to parallel quantum universes and we slide through it. Your world is different from mine, another realization of the same possibilities.”

“That is *Kauderwelsch!* [5] What shall a quantum be? A given quantity? But that can be any value!”

Now Ludvick was surprised. No quantum physics in this universe? Oh my!

“Sorry, I try again. The world, ourselves, evolution of life, human history, the visible and invisible universe is in a given stage and consistence as of today. However, it could be and could have been different. Depending on sometimes tiny changes in some details.”

“I agree, that is imaginable. So you are hinting at a technical possibility of sliding between the continuously many alternatives? By that machine?”

“Somehow, yes. Professor, would you explain me for what purpose you were building this machine and your experiment?”

“If I reveal that, you will not leave this establishment ever. Understood?”

“Still, I am curious. I did not intend to land here, but once I am here, I want to know this world.”

“Good. My project was to gain energy from the nothing. More precisely to make the energy accumulated in the curvature of spacetime for a useful drive

[5] Double Dutch!

to our machines. To make life easier."

"I imagine, it is a huge reserve..."

"Practically infinite! Since space and time and matter are continuous, also the forms of energy are, and then a curvature can be even infinite. In principle."

"So what is the obstacle? Technically, I mean."

"That we do not know how to go beyond a given value of energy density. In principle there should be no limit, but... as there would be a mystic wall to stop us."

"You mean as space would have undividable atoms?"

"A–tomos is a good old Greek description: un–dividable. But this contradicts the continuum assumption."

"It is not possible to abandon that assumption?"

"That would ruin our whole science!' the professor made a frightened face. 'Our universe, as a matter of fact, *is being* continuous."

"I accept this as your knowledge of to date. But how do you explain then, for example, electricity?"

"No problem. Electric, and for that matter magnetic fields, can be everywhere and at any time. They are as continuous as spacetime itself. We describe them with a bundle of continuous functions."

"But what about their source?"

"First of all, they exist without a source. The Hertz-waves propagate through the Aether, this is the basis of our radio technique and radio astronomy."

"I mean, there must be an elementary unit of charge, that you cannot divide any more."

"Why must it be? No fact in this universe makes us to think so.' Professor Grünstein squinted at Ludvick. 'Are you making jokes on my account?"

"No, Sir. I apologize. My earlier university studies had a great deal about carriers of such a finite elementary amounts and we made statistical models about it..."

"What a nonsense! Any value can be taken with the same right as elementary. It was like using a network of quadrats instead of the plane."

"Indeed, we did also that. We called it 'lattice theory'. It helped to make our computations more automatic and through that – easier and faster."

"Hmmm. A strange idea. But as a helping model, if it really, I mean *really* worked... could be considerable."

Professor Grünstein contemplated for about ten seconds. Then he triumphantly opened a new deal in this conversation:

"I know what. Let us take the entropy as an example. More generally probabilities and probability distributions." He went to the old-fashioned blackboard at the wall, opened its wings, and started to write formulas in the middle.

→ Discrete vs continuous PDF, page 140

"You see... a non-continuous statistical model cannot describe our reality,' concluded Professor Grünstein. 'A continuous function is not known by knowing its value only at discrete points."

Ludvick tried to remember his studies in analytical math. It occurred that this was not the whole truth...

"But excuse me, there must always be an $f_i = f(x_i^*)$ value for which the sum and the integral in fact are equal, as long as the function is smooth. As far as I can remember *my* math studies a decade ago."

"Young man!' lectured the professor, 'Of course I know that theorem. But one can never ever pinpoint, or calculate within a finite number of steps, that which is the corresponding internal x-point."

"That I admit, Professor. Still I perceive this as a strange difference to my universe, I mean not having quanta."

"Au contraire!'[6] responded the chief professor, 'We can have *any* quanta."

"Exactly that disturbs me. Without quanta in radiation energy, called photons, the total energy irradiated by a black body at a given temperature would diverge to infinity."

"Ah,' waved Grünstein with his hands angrily, 'there are no absolute black 'bodies' in nature. Admittedly, for some unknown reason, the emitted total energy is never infinite. But this is no wonder: we could not measure infinite energy."

"The whole universe must have been cooked, before we could talk," noted Ludvick.

The professor became angered. He changed his tone.

"We have to finish this crazy talk. Since you are an intruder, possibly a spy of

[6]On the contrary!

the Spaniards, I call the security force. Wait patiently here, this is the best for you."

Ludvick was surprised. He did not wish to encounter the rough men of this strange universe. *It must be a dream, it must be a dream...* He concentrated to wake up.

Darkness fell on him. Did he managed to switch his awareness by will? It contradicted to being in a dream. He opened his eyes: he was lying in his bed in the room assigned to him. The sky began to light up a little, first grey and then more reddish. A new day has begun.

Ludvick's report

At breakfast Ludvick appeared with a wrinkled face. Ken took this opportunity to pinch on him:

"Ludvick, you look as having had a hard night,' Ken told this with a very wide smile on his face, "did Ginny finally opened to you?"

"Ken, you know nothing. We, Ginny and myself, have very few in common. And that is all public."

"Ken, you do not know me. You have no idea about women's strength," hissed Ginny sharp and aloud behind Ken's seat.

"Please, have mercy on me,' replied Ken playfully. 'I was assuming something I should not have."

Ludvick did not like when someone else tries to steal the show. He announced firmly:

"I had a dream. The strangest dream I ever had."

Some more heads turned towards him. Ludvick started to like this research team.

"In my dream I was in a parallel universe. By the help of the mysterious machine in the cellar – at least that was what I felt."

"So how was it?" asked Rudolph encouragingly, from beside a huge mug of hot coffee.

"Imagine, no atoms in that world." These words created an immediate silence. *What? No atoms at all?*

"Did you dream of a totally empty world in darkness?" asked Fred with a curious expression in his eyes.

“No, not an empty world. Just a world of different nature. Without discreteness on the atomic level, no sign or knowledge of Planck constant, the whole science built on continuous functions as models.”

“How was then life and evolution possible there?” wanted Rudolph to know.

“Everything looked and felt like in our universe. There were people around, moreover scientists. They experimented with the machine downstairs. They just believed in no atoms.”

“Did not they have problems with experimental facts? What about black body radiation? Stability of atoms?” cited Santino the most well-known arguments.

Ludvick shrugged, “I do not know. I had a disputation with the chief professor there. The ’Ego Zeus’ of them, called Grünstein.”

“What an interesting idea!‘ claimed Ms Kunz by joining the group. ’You rather dreamed of Ego as any of us! Or for that matter of anybody else from your sensual life.”

“I could not direct that dream. Or hallucination. It wore me out at the end.”

“So, without any discreteness in their models, how did these dream-people make statistical calculations?”

“Oh, the professor told me that they accept counting models as approximations. Those just do not describe reality, their reality, at its heart. The true world is continuous: this is their ’continuity assumption’.”

“Hm, dear colleagues. This must have been a nonsensical dream. I think rather the opposite is true: continuous models are approximations to the discrete and countable reality. At least if we model money and economical inequalities...‘ Vilmano emphasized the last sentence. ’I admit, however, that the use of continuous distribution functions makes several formulas simpler. More elegant, if you wish.”

A cacophonic dispute was released at this point. Everyone had to say something, possibly immediately, and that happened just slightly asynchronous. Finally, Ego arrived and saw that he must govern the day’s work into new channels.

“Professors,‘ shouted Ego, ’silence please. I call your precious attention to our original project: modelling wealth and income inequality. Please stop this childish debate and let us gather in the discussion room with fresh and open minds.”

Ego marched firmly out of the dining room. Somewhat calmed, one after another followed. In ten minutes the common breakfast and dining room was departed.

Vilmano's supplement

This was the fifth day, and the mind workers started to show signs of fatigue. That showed in their rapid excitements and easier de-excitements, the light diversions of topics. Ego decided to take an iron hand on the agenda this day. He called:

"I think it is time to start with the discussion about models of economy. I call Vilmano to present us with some technical details."

"Good, I volunteer,‘ replied Vilmano, 'but only if I may first react to Ludvick's nonsensical ideas."

Ego capitulated. He waved towards the whiteboard with a tired gesture.

"Very well,‘ started Vilmano energetically, "let me first discuss the alternative description of probability distributions over countable and continuous sets. You will see that both the dynamics – describing the changes in time by mathematical means – and the statistical methods – deriving quantities with one-sided changes and hence achieving maxima – are modelled either by a continuous or by a discrete set of model states. To start with, I compare some solutions of discrete and continuous models."

→ Discrete vs continuous master equation, page 140

After having Vilmano's derivation presented, the research team showed quite a number of sceptical faces.

"So far it looks only as another notation, for practically the same thing," gave Ginny a remark.

"Well, indeed. This is even so for the H-theorem, the statistical counterpart of the second law of thermodynamics. It states that a certain quantity, constructed as a sum using the distribution, can only be reduced by the above equations," replied Vilmano.

"As far as I remember, this quantity is the negative of the Boltzmann entropy," inserted Ludvick.

"So please show us differences, not the similarities. For the sake of a clear distinction," asked Ego.

"That I can do by example. Then you see particular differences in the solution of the analog equations. I'd like to introduce the 'Local Growth Global Reset' model." Vilmano turned again to the board.

→ Local Growth Global Reset Model, page 141

After this brief introduction into the Local Growth Global Reset model, the research team became even more sceptical.

“This model is very simple-minded. You did not leave room for describing among others the diffusion,” criticized Rudolph.

“Indeed, the small-step process, ’local’ in the state space, is directed. Only growth is allowed. To counterbalance this, the LGGR model cares for resets to the ’zero state’. Still a stationary distribution over the half-infinite chain of states is possible.”

“What processes are described in such a construction?” asked Ken.

Vilmano hesitated, then gave an evasive answer.

“For example an income or wealth distribution is defined over non-negative values only. If you restrict yourself to a given currency, the units of it define the smallest possible step. Say one cent.”

“On the other hand the continuous treatment is a good approximation for millionaires,‘ noted Fred half joking. ’Please go on, let us see differences.”

“My first example is the simplest: the transition rates do not depend on the starting state. Here poor or rich have the same chances to earn one unit of money more or to jump to the total ruin; or for that matter just leave the data set.”

→ Constant LGGR rates, page 142

“These two solutions are Boltzmann exponentials, but with a different temperature parameter. Here is one difference,” concluded Vilmano his derivation.

“The difference actually vanishes if the reset happens much less frequently than the local growth,‘ noted Fred. ’I wonder whether the result is realistic in describing social inequalities.”

Several heads were shaking. Vilmano confessed.

“In fact, no. The exponential distribution may be very common in physics, but in economics it proved to be a faint trial. The favourites are the Pareto- and the log-normal distributions – just from fitting data.”

“Can your little model deliver those?” wanted Ego to clarify.

“Yes, it does. The principle beyond the Pareto distribution in this model is a linear preference for the growth: Whoever hath them will be given.”

“This sounds like a citation from the Bible. Gospel of Matthew,” noted Rudolph.

“Indeed, therefore its name is also ’Matthew’s principle’. The LGGR model incorporates this effect by making the growth rate itself growing linearly with the quantity of income."

Vilmano made a clear table and restarted by showing formulas.

→ LGGR Waring, page 142

"The result of your derivation... Is this the Pareto distribution?" inquired Santino suddenly interrupting.

"No, this is the so called Waring distribution,' replied Vilmano. 'The Pareto distribution emerges in the continuous model. For that I show you the general solution for the stationary – no more changing – distribution and then replace the linear preference growth rate into the formula."

Vilmano cleared again and wrote with well readable, block letters.

→ LGGR Pareto, page 143

"And this is the way how the LGGR model delivers the Pareto distribution," finished Vilmano.

"Some caveats?" asked Fred with a disarming smile.

"Caveats?' Vilmano wrapped his forehead. 'There can be many. First of all, not only this combination of linear preference and constant reset rates leads to a Pareto distribution in the stationary state of the LGGR model. But probably this is the simplest."

"And what about the log-normal case?" inserted Santino his question.

"Well, an easy way to generate it is a linear preference in the growth rate combined with a logarithmic reset rate. I show you."

→ log-normal, page 144

Vilmano closed his eyes and concentrated for a while. *"Too much of the formulas!"* he thought. Loud he said:

"A little more math is needed to see what is the freedom in choosing the rates and still arrive at the same stationary distribution. Let me discuss a few particular categories."

→ various rates with the same stationary PDF, page 144

"Amazing!' noted Fred. 'Then say the classical exponential distribution emerges also when both rates are linear in x, but well adjusted each to the other."

“I confess,” said Vilmano with a wry smile.

“So, after all this magic of the LGGR model, are we closer to the theory of inequality?‘ tried Ego to divert the discussion back to the original bed. ’Which distribution is less or more just?”

All started to talk at once. From this chaos some expressions like "H-theorem", "entropy" and "Gini index" emerged, just to sink again in the whirling sea of expert phrases.

“Doctors and Professors, we have only two days left to comply with the original research goal. I close this session and hope to make a new start tomorrow. You will not want to dissatisfy our sponsors, do you?”

By these words the afternoon session was dissolved. The researchers streamed outside to the fresh air in small groups of two or three. Some went alone. The sky was covered by dark clouds, anticipating bad weather for the coming night.

Chapter 6

Is it Common to be Rich?

The clouds did not lie. Darkness fell this day earlier than usual and deep roars were heard, lightning blinded and snapped, the wind became strong, and at the climax of the tension a heavy rain started to pour. The lights in the castle flickered a few times, but did not go out entirely.

The guests could not sleep. Most of them gathered at the bar, but no singing that night. Ken, with a glass of whiskey in the hand, approached Ginny in the corner. He looked at her for a while, and then started with a wry smile:

“Ginny, I know you do not like me... but I would like to consult you on a strange issue. Related to the machine downstairs.”

Ginny raised her head a little. She suddenly felt a warm sympathy for the guy, standing there, wanting something, and not knowing how to start. She gave a hint of help:

“You mean a novel secret text passage to discuss?”

“No, something bigger. I have tempered with the machine.”

“And that was not a dream this time?”

“Uh, it was not a dream before, either. Or all this around us is a dream, too.”

Ken took a seat beside Ginny, and continued with a low voice.

“Hidden in the back, there is a small plate, behind which cables are connected in a matrix of holes. I have realized that some of them were loose, and it looked as if originally they had been plugged differently.”

“How is that?”

“Well, the dust on the cables, it was wiped on some. These few also were spanned much more than others. So I have exchanged them back."

"With other words you did blind chance experiment. What did you expect?"

"Believe me, now the whole thing looks more natural. *It is repaired.*"

"Good, let me try now my chance. Would you lead me there, please?"

Ken obliged. He was proud of his achievement. And finally he found something, which would impress Ginny.

The two went down through the hidden entrance, this time Ken knew the code already. He made light with his mobile phone, and no alarm disturbed their advance. In the underground lab they entered, and Ginny seated herself in the machine. When Ken wanted to take the side-seat, she stopped him.

"No, please, let me do it alone. I do not want witness in my dream."

Ken shrugged and retreated. Ginny switched on the main stick and she felt falling into the darkness.

Ginny's dream

Ginny woke up in darkness. It was cold and windy under the castle, with unpleasant odours. It felt like not only another place, but also another time. She stepped out of the machine and tapped in the darkness to find the lab entrance. A wooden door was there, it opened with a loud jarring moan.

In the corridor, cold and wet, some small animals run across. She suspected, rather than saw them. Ginny advanced in the direction where she guessed the castle with its warm and friendly rooms. The fifty or so feet distance felt like an eternity. Finally, she arrived at some stairs, leading upward, to another massive door.

She could not pull or push, did not find a clink. How to open it? Desperately, she kicked on the wood. Then the door flung suddenly open. However, not she was the motor: two crude men run in, wearing some shabby and dirty cloth, bringing a stinky torch. They ignored Ginny and marched through the corridor. She used her chance, and slid into the castle.

The antechamber was much bigger, than she remembered. Also the furniture was different, richer, with a lot of deep red and gold upholstery, a Tudor style ambient. Also many doors opened to corridors and smaller chambers. She discovered a huge mirror, and what she saw, really surprised her. A rich lady from the court of Elizabeth I appeared in the mirror. A brocade gown in deep green, golden insets in her hair, her reddish mahogany strands carefully arranged into fractal structures of spiralling waves under a thin and very fine net, true pearls around her neck with a shining white, a very long skirt reaching down to the floor and being decorated with a fine lace made her appearance breathtaking.

A giggling was heard from the right. Three young women approached enjoying themselves in a conversation. They smiled almost all the time. Their way of going let the hips swing a little, the long and heavy gowns giving a certain musical noise. When they passed Ginny, they bowed, with short and stringent moves. Ginny smiled and bowed, too, as well as she remembered from the movies seen in her teen age. Suddenly she felt a chill. An elderly woman, dressed in black, stood close to her.

"Lady Virginia, Her Majesty is expecting you in the Pearl Chamber."

Ginny got frightened. She turned to the voice with a determined move.

"At which time did she ordered?"

"Immediately."

"Then go ahead, show me the way."

If the woman was surprised that Ginny did not know her way to the chamber where the queen received her ladies of the court, she did not comment. Ginny followed her, and in five minutes they stood at the massive, yet inviting door of the Pearl Chamber. The woman nodded to the service maid and then she announced:

"Lady Virginia at your service, Majesty."

"Let her in."

Ginny entered to the richly decorated chamber scented with heavy fragrances and some flowers as decoration. The queen sat near to a colored window, in a fireplace a friendly fire spread warmth. Elizabeth waved her to come closer and with another wave of her hand dismissed the maid.

"Lady Virginia, I have to talk to you about serious matters."

"And that will be, Majesty?"

"You are already over the standard age, when ladies of the court are usually married. We want to reciprocate some of the services you and your father have done for the Crown."

Ginny flushed.

"I beg your pardon, Majesty. I have not found yet the nobleman, worthy enough."

"What do you think of Lord Kenson?"

“That old goat, Majesty? He would get a heart attack on our first night!”

“No. I mean his nephew, Lord Kenneth Kenson. A young noble, was introduced to me officially two days ago. He is new at the court, and as it seems, does not intend to stay long.”

“But, Majesty, I do not know the young groom. What if I cannot love him?”

“No matter. He will earn a big fortune. You can be the wife of England’s richest man. On the top of that, he is young and handsome. Some other ladies would be delighted to associate with him. A few of them even without a marriage.”

“And, if I may ask, what did you mean by not staying here long? Is he ill, cursed with a deathly disease?”

“God save us, no! He has plans to settle over to the new world. In the name of our Crown, of course.”

“Then I will leave London with him, and ship to America? Oh, Majesty. I am afraid it would be too much of honour, that I do not deserve.”

“You ungrateful! I make thoughts of your fortune in these trying times, and you dare to refuse to cooperate?”

“Mercy to my poor head, Majesty. I beg your highest permission not to marry. In change, I will listen what trouble drills you.”

“You were always intelligent, Lady Virgina. Yet I cannot hold you longer at the court. The young ladies of noble families need their places, too.”

Elizabeth deepened in her thoughts for a minute. Then she sighed.

“It is not at all easy for a woman to bear with a man’s job.”

Ginny could not agree more. She faintly remembered her another life in another world, in another universe. She gave a last try to catch the queen’s attention.

“But women have a different way of thinking, than men. It can help.”

“Lady, to rule a kingdom, not to mention an empire, is hard work. We need to keep our balance while getting forward. And many are ready to cross our intents.”

“Like who?”

“Like the Spaniards. They want me to marry a catholic noble, in order to usurp Britannia. I cannot let it happen. But what if they attack us? With their Grande Armada?”

Ginny remembered her studies in history. Yes, the Grande Armada was beaten and diminished exactly when they attacked Elizabeth’s England. But that alone would have not sufficed. The weather was on the English side, too. And Francis Drake, who had experience with Spanish ships already, who knew how to sink them, and who was included into the military council of Lord Howard, the commander of the English fleet. The English ships in general were superior in manoeuvres, faster and more flexible. Also the supply lines were short for them. No doubt who won that great battle.

“Majesty, the English sailing is ahead of the Spanish. More traders, more sailors, more privateers on our side. And once the Armada is in the narrow channel, it will be no use for them being many. Au contraire!”

“Look at that. Lady Virginia is an expert in men’s science,‘ Elizabeth smiled wryly. ’Spain is the leading power among all nations, right now. What do you know about?”

“I beg your pardon, Majesty. But I do believe that England will win, no matter who attacks us on the sea. In my dream it has happened already.”

“I will not risk my kingdom due to a stupid dream of a girl! We make peace with Philipp.”

“Yes, Majesty. Please do so. Although, as far as we know the Spaniard king, will they keep to their word of peace?”

“Stop here! You are now blaming royalty!”

“Have mercy on me. I should have said, when God is with us, none is against us.”

“That I cannot deny. But enough of crazy talk. You are dismissed for now.”

Ginny bowed deep, and started to back off. She felt dizziness again, darkness fell on the world. She felt going through empty space and waked up sitting in the chair of the mysterious machine. Ken was shaking her.

“Ginny, Ginny, are you with me? Please, open your eyes!”

She opened her eyes and smiled faintly.

“I am tired, Ken. Please let me return to my chamber.”

Ken noted the use of the expression "chamber", old-fashioned and pompous, yes, in fact magniloquent, but he was too much upset to make a comment. The two silently returned to their rooms.

Fluctuating dimensions

At the next morning the late autumn sun was painting the colors in the dining room shinier than otherwise. The researchers gathered slowly, one after the other, to take a breakfast as well as to continue their disputes at the white table. Ginny, Ken, and Vilmano sat at the common table. All of them were in a splendid mood.

“Good morning Ginny, Ken! You look as having had a resting peace of a good sleep lately,” opened Vilmano the table-talk.

“Oh, Vilmano, indeed I had a strange dream last night. You may not want to know...” told Ginny in a musical, dreaming tone.

“Come on, we are always interested in your dreams...” smiled Ken, somewhat brighter, than his words indicated.

“I have dreamed of being at the court of Queen Elizabeth.”

“It sounds like a teenage girl’s dream in London...”

“No, not our queen. The other one, Elizabeth I. Daughter of Henry VIII and Anne Boleyn, the last Tudor on the English throne.”

“A historical dream, then. What have you done there?”

“We had a conversation about my marriage to be, and about the forthcoming invasion of the Spanish Armada.”

“How did you come to that? Even if it is a dream, it must be difficult to get to the royal ears...”

“Well, in my dream I was a lady of the court. An intimate friend of Elizabeth.”

“And did you foretell her future in that altered past?”

“I tried, but she did not believe me. Than I was dismissed, and I woke up.”

“So we learn that human behaviour is not only elastic, sometimes it is stubborn. Have you experienced some of the others? The society behind? Riches, poor?”

“No, not really. It seemed to be common to be rich. At least judging from the cloth everyone was wearing. And from the furniture.”

Vilmano deliberated for a while. Then he told seriously:

“A proper topic for today’s discussion: is it common to be rich? By other words, how rare is rare? How big proportion of the people is over and under the average?”

Ken shrugged.

“If you mean their height or weight or other bodily properties, then I guess ’over’ and ’under’ are about equal. The distribution is symmetric around its maximum, and the average is close to the position of the maximum. Average value, most probable value and median are close to each other. That is the reason for not distinguishing those in the everyday speech."

Fred joined to the table. He slurped his hot coffee a bit and then intervened.

"Are you referring to the *central limit theorem*? Anything, formed from a big number of independent random measures additively, comes close to the Gaussian, the bell-shape distribution?"

"Not only,' answered Ken. 'In fact it is more interesting, when it fails. I mean the Gaussian result."

"Certainly, when one of the conditions of that theorem is not valid." Fred did not see the depth in the question.

Ginny came to rescue.

"Fred, we were discussing whether it is common to be rich. Certainly the answer depends on how common, and how rich, but you see, here are two quantities playing with each other: the cumulative rich end of the population and the cumulative wealth possessed by them."

"As I have shown in my talk on the Pareto rule, a few days back," inserted Vilmano.

That was the point when Santino and Rudolph joined the table.

"I have an interesting example from physics when the Pareto distribution emerges. The driving mechanism is the fluctuation in the number of dimensions. I offer a compact lecture about it to start this morning session."

"We take it, only because it deals with the Pareto distribution. You may demonstrate to us that this PDF is also common. And it makes more rich people than the exponential PDF," added Vilmano.

"Very well, let us go over to the lecture room," invited Ego the team from the exit door. He wore his white suit from the first day, perfectly ironed and cleaned.

The team gathered again in the lecture room, and indeed Santino gave a short briefing about his model using fluctuating dimensions.

"To make it simple we consider an ideal ensemble of actors, like atoms in an ideal gas. They share a huge quantity, energy in the physical world, which is distributed in their motion. This quantity, the motion energy of independent particles, is additive," started Santino his talk.

→ Fluctuating Phase Space Dimensions, page 145

"As you realize easily from this derivation, the negative binomial distribution, short NBD, in the number of particles sharing a given total energy is responsible for the Pareto distribution in the individual energy of a single selected part," finished Santino his math show on the whiteboard.

"Wow, this is impressive,‘ claimed Ego, ’but can you tell us more about this mysterious NBD distribution?"

"Surely, I can. Best we do it by a comparison with the more familiar Bernoulli distribution and Pascal triangle. I shall start with a general power of the sum of two quantities."

"Santino, be careful not to sink to the elementary level of small children," tried Fred to cheer everyone up with an intentionally humorous remark.

"Do not worry, Fred. I dig only as deep as to find the treasure."

→ Bernoulli and Pascal, page 146

The presentation of the familiar Bernoulli distribution and the related Pascal triangle arranged from the values did not impress much.

"Known since the seventeenth century," noted Rudolph.

"And still true in our century," countered Ginny.

"I just do not see how a Pareto distribution arises from this, since I do not know the negative binomial distribution. A negative binomial ... is it divergent? Or zero?" drilled Ludvick unexpectedly on the theme.

"Let me continue,‘ intercepted Santino further comments. ’The name ’negative’ comes from the power being negative, not the coefficient."

"But the role of all these negatives is at the end positive," tried Ken to insert a dry joke into the discussion.

Santino turned to the white board.

"We do the same as before, only for negative powers. Look!"

→ NBD version, page 146

Once given the negative binomial distribution formula Santino wanted to triumph. He started to encircle the final result.

"And where is the Pascal triangle here? I cannot see..." interrupted Ego. Some of the professors nodded, Fred and Ken smiled devilish.

"Oh that is easy. You just have to change your point of view by 45 degrees."

Ken and Fred tilted their heads in opposite directions. When they realized it, they smiled, and a small laughter developed in the audience. Santino laughed with the others, then he turned again to the board.

→ NBD Pascal triangle, page 147

Santino summarized his examples on the negative powers of two-part expressions, the binoms.

"Facit: the rows of the Pascal triangles are the cross-diagonals of the NBD expansions,' summarized Santino. 'Here you see (1), then $(1, 1)$, then $(1, 2, 1)$, followed by $(1, 3, 3, 1)$, next by $(1, 4, 6, 4, 1)$, finally in the above expansion by $(1, 5, 10, 10, 5, 1)$. The full Pascal triangle is there, only rotated somewhat."

Fred raised his hand. Santino and Ego nodded towards him, *"Speak up, please!"*

"I wonder what is the reason behind. For the original Pascal triangle we had a formula between coefficients in the K-th and $(K - 1)$-th power expansions. Can you show a similar formula?"

"Come on, Fred, surely one can do it in a few lines. Please do not divert us from the research goal we are here for!" Ego shook his head. *"With formulas we cannot satisfy our trustees..."* he thought. Loud he made only a suggestion:

"Dear Professors, et cetera, let us concentrate to the real world again. Please educate me on examples, where it is more common to be rich than mother nature would suggest it by her atoms."

Vilmano injected his remark:

"The Pareto distribution! It tells that you may belong to the richest twenty percent even with a wealth below the average! Down to 87 %."

"Every non-Gaussian or non-exponential in another quantity, usually related to the square of the previous, will predict more common high values than most distributions, formed by independent random changes. We call those 'fat tail distributions'," added Fred.

So the discussion turned over to fat tail distributions.

Fat tail distributions

Fat tail is a strong metaphor. Compare the peacock to a sparrow or to a robin, a fat tail is beyond the bare necessity. Its prevalence among living creatures, first of all birds and reptiles, demonstrates that the "survival" is not only for the "fittest", but rather for those who pretend their fitness most successfully. They shall leave the most descendants behind.

A fat tail is not helping the flight, it announces a drawback. *Look, even with this hindrance I am here and living and want to mate!* And it works – otherwise no such thing like a peacock's tail would exist.

The fat tail in statistics is different. It occurs in the picture of distributions, it is more frequent than it would seem at the first glance. It is of course only a slang to call it "fat tail". The precise meaning of it refers to a distribution tail which is above the most common bell-shape, the Gaussian, the falling exponential of the square. Talking about functions, such PDF-s at high values of the variable run over the exponential function. They often behave as power-laws.

First Vilmano emerged to lead the theme.

"Fat tail distributions, typically power-law tailed ones, we encounter quite often in wealth and income distributions. The Pareto-law itself fits asymptotically to a power law. On a logarithmic plot it curves up relative to the line of the exponential."

"You also mentioned the log-normal distribution before. Do they also run over the exponential, better to say, over the Gaussian?"

"Yes, indeed. The logarithm bends below the linear asymptotically, so replacing the variable by its logarithm in the exponential runs above the original."

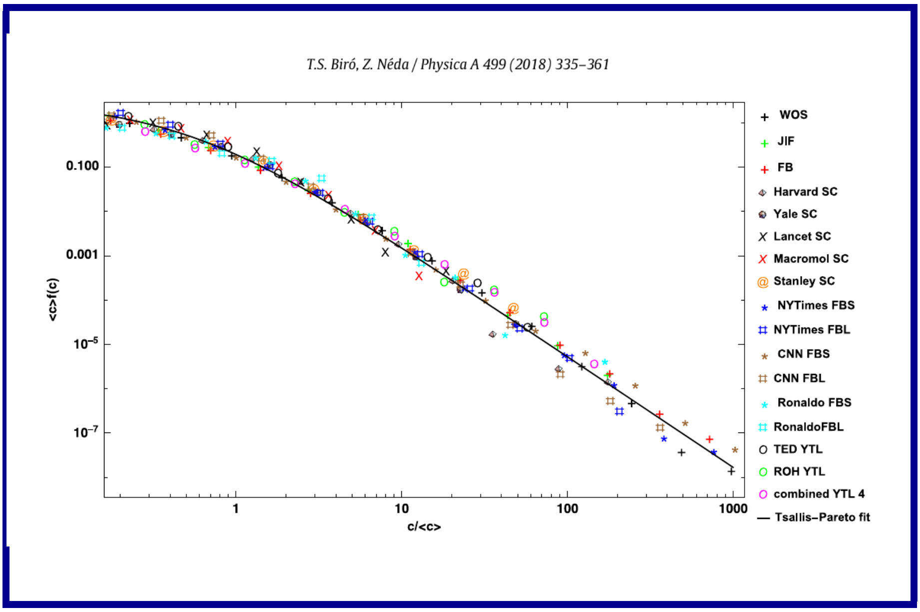
Ego wanted to shepherd the discussion closer to real life.

"Examples! Show examples of fat tailed distributions, gentleman. Please."

Fred raised hand.

"Please, allow me to bring further examples for the Pareto distribution. This distribution *is* definitely fat tailed."

He put his memory stick into the proper slot on the table and the following graph has been projected.



Fred provided explanation to the figure:

“This is from an open source publication available on the internet. The authors gathered popularity data: Citations of scientific articles, that of whole journals, citations of publications from renowned universities. Beyond that You Tube and Facebook likes and shares of individual and institutional pages were provided, as close substitutes to citations.”

“Sorry, this legend is full with abbreviations...”

“Indeed. Well, WOS stands for Web of Science, it counts citations of scientific papers. FBL and FBS for Facebook likes and shares, respectively. YTL for You Tube likes, ROH for the Royal Opera House in London. TED and CNN, I suppose, are known... Uh, SC for scientific citations, NY for New York...”

“So what shall we learn from this?” wanted Ego to hear.

Fred nodded, “The important point is on the axes of the graph. First, both are logarithmic, so the straight line for large arguments represents a power-law tail, a fat tail of the distribution of these data. Second, they all fit by a single black curve, plotting the Tsallis-Pareto distribution, but only as a function of the ratio of popularity citations to their average. A wonderful scaling law!”

It took quite a few seconds before the research team started to like this picture. But then, smiles and sparkling eyes appeared, offering the nicest of all possible rewards for the speaker. So at the end, he was smiling, too.

“I am almost convinced by this. Yet, what is the mechanism? What makes this distribution so similar to the wealth and income distributions?‘ contemplated Vilmano further. ’Where is the reset process in citations? Once cited, forever cited, I would think.”

“Such a question may come only from the sharpest minds!” exclaimed Fred.

“Certainly we would need to read the original paper for learning about the reset process. Briefly I can only say, that the scaling shows us, that possibly an exponential growth of the average itself causes a diluting, similar to the reset term.”

“Oh, that I have also experienced earlier, when studying exponential growth processes which were competing with each other,” noted Ken. He jumped out to the whiteboard and hastily wrote a few lines.

→ Exponential dilution, page 148

“This effect is the exponential dilution of the data space,” repeated Ken his conclusion once more.

“So what else can be described by fat tail distributions beyond wealth and popularity? Is something more general behind it?” asked Ginny the final question.

The men shrugged and made wry faces. Nobody knew an answer.

“By having as many examples as you wish, we cannot conclude to a general law of nature,‘ warned Rudolph. ’Reset, exponential dilution of the sample space or any other, carefully designed value-dependent rate of change of the occurrence frequency of that value, may work in the background.”

“We have also learned from the general solution of the LGGR model that a given PDF does not specify the rates, only their combined expression.”

“Network connectedness, degree distribution is also power-law tailed. Sometimes exponential. For the fat tails always a preferential connection strategy acts in the background,” added Fred to the discussion.

Ego sensed the dead end this discussion was heading to. He declared a coffee pause, and the team members rose from their seats. Ego used the change for disappearing from the lecture room. He went to his private study room, since he was expecting an important call in a few minutes.

Captain Ego

Ego was just a bit nervous. Quite a bit. He switched on his hardware protected laptop, and opened the icon leading to a secret channel. Only those could communicate on that channel, who had the small device attached to the computer.

A videophone window opened automatically, and was waiting for being called. Ego ordered his face for looking neutral, although he felt the growing tension inside. As often, the calling party was late. The privilege of important people.

Finally, the call signal disrupted the tense silence. Ego reacted, he clicked on the "accepting call" button. Instead of a person, only the emblem of the OFS appeared on screen. The voice was artificial, unrecognizable whether a human was speaking or not.

“Ego Zeus, your report is due. What are your experts saying about manipulation possibilities of wealth inequality?”

Ego swallowed and then started with a subdued voice, not heard lately from him:

“Sir, we need a little more time for the final conclusion...”

“What?! Are you trying to bargain with me?”

“No, Sir. Not like that. I just mean that... well... we were working hard since weeks, yet the experts are gathered only very recently, and...”

“I do not want to know details. Those are your problem. I pay you to deliver results.”

“Yes, Sir. We are already close to that. I beg for two more days to formulate the expert conclusions, and you will have the desired report...”

“Ego, Ego. You must know me. In forty-eight hours from now you will deliver. Otherwise bear the consequences.”

“Yes, Sir. Ego out.”

Ego disconnected the line. He realized only now how heavily he sweated. He made a fist for not seeing his fingers trembling. He took a deep breath. And then one more, slowly blowing out the air. He smoothed his face to stop or to prevent tickling. Then he took a glass from the drawer of his personal desk and drunk. One, two, three sips. In a few seconds he was back to normal.

Are these scientists of any use for us? Ego was deliberating in thought. Words from Goethe came to his mind: *"The mathematicians are like French. Anything you tell them they translate it to their own language, and from that moment you do not recognize it anymore."* He expected at least some clue for smart controls for influencing the inequalities plaguing and destabilizing the modern world. Actually making the rich men’s coffee sorrow by the shear existence of the poor. It cannot be denied indefinitely long.

What to do, now? Try to court the professors? Be harsh on them? Threaten them? Would it deliver results? Ego sunk in his thoughts. Long minutes passed, and then suddenly a harmonic music alarm signal came from his laptop. Ms Kunz was calling.

“Chéf, it is time to talk to the team. They are waiting for you in the discussion room.”

“Thanks, I’ll go.”

Ego closed the window, switched off his laptop here, locked it in the desk drawer, and before leaving the private study he armed the alarm. He approached the discussion room with resolute steps. He stayed standing in front of the gathered team, and started a speech.

“Professors, Doctors, Engineers! We are at a critical point. Our sponsors have lost their patience. We have to deliver results in two days, more precisely in forty-seven hours and twenty minutes from now.”

The silence deepened and tensed in the room. All were listening Ego. He proceeded.

“Starting tomorrow morning the only topic of discussion will be the measurement and manipulation of income inequalities. Nothing else. No entropy, no informatics, no stochastic models. We have to compose a report on the topic *in layman’s terms*. The math behind could be an appendix, but will not be read. Most probably not.”

“Do you want us all write a report? Or just a common one?” asked Rudolph.

“Certainly, one common report suffices. You may divide the task among yourselves, but it should become a united result. Do not sleep and dream, think on the report by now.”

The researchers looked each at the other. None objected. Also none said "Hurrah!" They kept their opinions this time. Then Fred started to share a plan:

“I think the best will be, if Rudolph writes the introduction. Santino and Ludvick the appended technical background. Vilmano the economics part. Ken, Ginny and myself, we would support you with data, presentation tables and graphs, and by reading texts and suggesting corrections.”

“This plan is acceptable. Let us start tomorrow morning right after breakfast,” suggested Rudolph. Santino and Vilmano nodded, Ludvick shrugged with a face meaning OK, Ginny and Ken kept silent. By this the work sharing was done.

Having a work plan for tomorrow the researchers relaxed for tonight. They departed the lecture room, had some rest or recreational sport for a short while, and then rejoined at dinner. Again jokes and songs crowned the day, led by the Latin tempered colleagues.

Rudolph's dream

The night did not bring relaxation. At least not for Rudolph; he volunteered for writing the introduction to the final report. This is the most sensitive part, here will the reader decide whether to read further or not. It is not advised to bring details here, except the most attractive one, if any. The introduction has to be an exciting promise.

Some use an introduction for "motivation", and yet fail. A simple repetition of that what the title and a short abstract already revealed is too weak. A conscientious list of what will follow, even the most precise one, is not telling enough. The introduction should suggest why is the opus worth to be read. And, instead of arguing, it is much better to show the mesmerizing core of the message directly. If there is any. A gospel, an *Evangelium*: a message from angels.

In all other cases it is better to skip the introduction. Rudolph could not sleep due to his own thoughts, he became increasingly excited and increasingly tired at the same time. He arrived at a stage on the border of conscious and unconscious. Then he saw a blueish light flashing through the door of his sleeping chamber. As if in a dream, he rose, and went out. He still wore his daytime suit of jeans, shirt, and a pullover. And of course shoes on.

Before he realized what he was doing, he found himself in the dark corridor beneath the castle. On his direct way to the lab with the mysterious machine. By his entry, damped lights went on, shedding enough to seat into the machine's driving cabin. The background AI sensed him, and the thin silver threads attached to the back of Rudolph's head. A new illusion started.

It was summer, a pleasantly warm afternoon. He was sitting in a garden, at the middle of a longish table, full with cakes and coffee mugs. Also some fruits, early apples, cherries, and raspberries in small dishes. Whipped cream in great amount. A German afternoon party, somewhere near to Frankfurt, in a big garden. *Ein angenehmer Nachmittag.*[1]

Rudolph felt light hearted and happy. About two dozen people were around, talking and laughing, and were particularly many children. And he felt a presence, that he did not feel in a long time. Rosalinde! She was sitting on his right, close to him, that he could breath in her scent, feel her touch and hear her warm contralto voice. She was alive.

Some of the younger people gathered now, and started to sing: the German

[1] A pleasant afternoon.

version of the Happy Birthday song. It was him, who had the birthday. He stood up to thank the presentation. By this he felt unexpectedly dizzy and weak, so after a faint *Danke euch Alle*[2] he splashed back into his chair. Rosalinde pressed his hand.

"Dear Rudolph, this is a great day! You are one hundred years old now. Or should I say one hundred years young?"

This coquetry was typical of the beloved Rosalinde. He turned towards her and smiled. Rudolph's happiness was almost unbearable, his heart beat so strong, as if wanted to jump out of his chest. Luckily, it did not happen. He asked Rosalinde:

"Who are all these people?"

"Oh my dear. It is our family. We have three daughters: Julia, Anna, and Kerstin. Their husbands, Udo, Carsten, and Chris, are all around. Then our grandchildren, Alex, Justin, Ilona, Margo, Steffl, Eros, Gunda, and Ariane. They are grown-ups, with their own families, only Ariane is still unmarried. And our nine grand-grandchildren in their age between ten and twenty-two. And our first grand-grand-grandson, Albert, the small baby over there."

Rudolph was wondering. This was his most beautiful, and at the same time, most aching dream. Having Rosalinde and a big family. Being happy together, not minding big problems with masses of folks and with the universe. Life, at its roots, presenting herself vividly and manifestly.

A young blondine with shoulder long hair, wearing a white dress approached Rudolph.

"Uropa![3] Would you cut the birthday cake?"

A huge cake was brought to the table in front of Rudolph. It was covered by a white fondant and the number 100 was formed large on the top of it. The number flickered with a bright golden color, probably LEDs were built in, or something similar. Or a chemical trick was responsible. Nanotechnology went already far.

Rudolph took the traditional silver knife in hand, and cut a piece out of the cake. Everyone cheered.

He took that piece on smaller plate, and started to taste it with a little fork. The cake looked strange. It fell into smaller pieces, each so big like a worm, looked like a larva of the May bug. And it tasted so. His stomach contracted, as if Thor's hammer had rushed into it. He looked up with a red face.

"How do you like it, Daddy? It is the healthiest food with the highest pro-

[2] I thank you all
[3] Great-Grandfather!

tein content we could buy," said an ageless looking woman, possibly one of his daughters.

Rudolph frowned, "I would have preferred a sweet, sugary cake like in the old times... What about the Black Forest cake?"

"But you know, that is unhealthy. You would be dead by now due to diabetes... by eating traditional cakes."

"Perhaps so. Nevertheless, a long life without gastro-pleasures... is like a repeated movement in catatonia. Though, I am happy now, I am happy here... with you all."

Rudolph smiled at Rosalinde and his thoughts raced back to their young years. At the same time, he felt an increasing distance from the party. Like awakening from a sweet dream... Instead he slept even deeper without dreaming. In two hours the next morning came, and Rudolph found himself in his bed, everything looked normal. Exactly, this was abnormal.

Chapter 7

Inequality measures Complexity

As Rudolph gained back his consciousness, the feeling of having an unfinished task returned. The introduction! First of all, we even did not agree on a structure of the common report. His thoughts raced. What shall we do?

We had a project to understand economic inequality. We are far from a complete answer about the reasons behind, the dynamical laws and of any reliable prediction. On the other hand, we have brought together and discussed mathematical tools and general statistical laws, which might be related to a solution. We have identified the Gini coefficient, as a useful measure for an overall inequality. We have brought the entropy and information measures into attention. And the general laws describing its unavoidable growth in spontaneous processes in closed systems.

Certainly, the society and economy are most probably open systems. But how much open? How huge is the environment offering reserves? Not infinite. Every aspect is finite, and it makes the mathematical description much more involved than in the classical, innocent models of thermodynamics. We have to face entropy formulas, instead of a single chosen one. Even a given family of formulas, described by a few parameters, might not suffice. What measures complexity best?

Rudolph sighed. Let us meet at the breakfast and work. First we need to go through the math appendix of our report: we need to see clearly the formal connections between the Gini index and entropic properties in general. Vilmano? Or better, Fred, can be the first speaker today. Let me coordinate this program with Ego.

Gintropy

Fred was thinking over this offer to start the day's lectures only for a fraction of a second. Then he refused:

"Vilmano will be much better to start. From today we talk about economy, not about math."

Ego nodded, and asked Vilmano to start.

Vilmano went to the board and started his talk *in medias res*[1].

"Before talking about gintropy I have to explain the Lorenz curve. Among numerous Lorenzes, this one is named after the economist Max Otto Lorenz, who had proposed a special graphical view of income and wealth distributions in 1905. There are versions of this view, I show you the one which is closest to define a quantity – the gintropy – with entropy-like properties."

→ Lorenz curve, page 148

"The curve $\overline{F}(\overline{C})$, defined as I have shown, always run over the diagonal $\overline{F} = \overline{C}$. That makes the simple definition of the gintropy as their difference to a quantity with a definite sign. I show some proof."

"Hold on, please!' Fred intervened. 'Let us first study your picture."

"Yes, it is neat!" claimed Ginny.

"What about the area between the curve and the diagonal?" asked Ego.

"Oh,' responded Vilmano nonchalantly, 'it is exactly the half of the Gini index."

"Joker!" commented Santino and Rudolph as a choir.

"Just follow my hand," suggested Vilmano with a juvenile smile.

→ Lorenz curve area, page 149

"And following the definitions of the cumulatives, this doubled crescent form area is just the Gini index."

Vilmano let the air out, and started to return to his seat. But Ego intervened:

"Could you remind us why it is so?"

"Very well. I try to summarize it briefly."

→ Gini index as area, page 150

[1] "In the middle", Latin, used for an immediate discussion without introduction.

“Nice! But what do you call *gintropy*?”

Vilmano did not realize who was asking. Nevertheless, he felt that it was the time to answer.

“The gintropy is defined as the difference of the cumulatives, or as the deviation of the Lorenz curve from the diagonal. In some sense it is the density of the Gini index over the area which describes the inequality in one number.”

“But why its name ends to ’tropy’?‘ wanted Fred to know. ’I do not see anything reminding to entropy. Or to any other ’tropies’.”

“Well, indeed, the gintropy as the deviation from the Lorenz diagonal shows mathematical properties similar to entropy density. On the top of that in some cases even the formula expressing this quantity as a function of the cumulative population in fact mimics entropy formulas.”

The grimaces in the audience talked a saga. How could this be true? Vilmano felt that he had said too much suddenly. He stepped back.

“First I show you the formula for this definition and its nice properties.”

And he turned again to the whiteboard.

→ Gintropy properties, page 150

“Funny,‘ noted Santino, ’this means that your gintropy decouples from inflation, from the growth of the average income.”

“But that alone would not warrant to use a word like the classical ’entropy’,‘ interjected Rudolph. ’Do we have a *second law of econophysics?*”

Vilmano understood the question well. But there was no answer, since economy is not really a closed system. Instead he turned his back to Rudolph, and started to write more formulas, with meager comments.

→ Further gintropy properties, page 151

“I would like to summarize. The gintropy is the deviation of the Lorenz curve from the diagonal. This is its definition. Then it provides the following features:”

1. It is never negative.

2. It is zero only at the extreme poor and extreme rich points.

3. It has its maximum exactly at the average income.

4. It is overall convex as a function of any of the cumulatives.

5. Its integral over the cumulative population equals to the half of the Gini index.

6. Its integral over the cumulative wealth also equals to the half of the Gini index.

“As you see these properties also reflect some entropy-like behaviour. This is one reason for the expression *gintropy*.”

“Most promising is the definite convexity of the gintropy – cumulative rich population relation,‘ sentenced Santino. ’Its graph is similar to the binary entropy.”

“Indeed, it is high time to see some examples for this function,” claimed Ken, fixing his eyes on Ginny. Ginny, as usual, showed her cold shoulder.

“OK, we have everything prepared for that. Let me try first,” offered Rudolph. Vilmano welcomed the change, and Rudolph made a clear board. Then he started with his pleasant baritone.

→ Natural gintropy, page 152

“So having an exponential distribution of income, the gintropy relates to the cumulative rich population exactly so as the Boltzmann entropy to the probability density,” triumphed Rudolph.

The team wondered for a while. Then a new challenger came.

“Now I am on the agenda,‘ called Santino vividly. ’I calculate gintropy for the Pareto distribution. What we have seen so far is just a particular case of it.”

→ Capitalism gintropy, page 152

“This result is nothing else than the Tsallis entropy formula, using the q-logarithm times the value, albeit mutated to a gintropy formula!” concluded Santino with a bright smile.

“Typical men!‘ noted Ginny with a little bitterness in her voice. ’You start with the complex, not minding the simple. What about the communistic case? When only one single income is allowed, and the PDF is singular?”

“That is *really* trivial,‘ cooled Fred her momentum. ’If only one value is allowed for, that is also the average: $x = \langle x \rangle$. Here the gintropy is constant as a function of the cumulative rich and its value is zero, as well as that of the total Gini index.

I suggest to look for something more realistic, say, a two class society. Like the real socialism."

→ Two class gintropy, page 154

After presenting his derivations, Fred turned towards the audience and straightened himself.

"A brief summary: the gintropy function is constant in the interval between the cumulative numbers at the edges, i.e. for the lower and upper incomes a and b. Outside this interval the gintropy is zero. The average income and the Gini index also depend on the relative weight of the classes, w fraction with the low, $1 - w$ fraction with the high income."

"What do we learn from this?' asked Ego. 'What is actually the Gini index in this two class model? How does it depend on the parameters?"

Fred shrugged, and turned back to the whiteboard.

"We can easily obtain that, it is simple algebra."

→ Two-class Gini index, page 155

After underlining the last expression for G in the two-class model, Fred added some explanation.

"Well, this case includes the communism for $b = a$, if there would be no difference between the incomes. In fact then the two classes would merge. In this case the Gini index is zero, and the gintropy is all over zero."

Vilmano added at this point his remark:

"In the other extreme limit, when $a = 0$, meaning that the low income class is actually a zero income class, we model a society with real proletarian and a probably narrow ruling class. Then the formulas simplify, and e.g. the Gini index will be equal to the proletarian fraction, $G = w$."

Ken raised his hand. Everyone looked exhausted already, but the better part of manners ordered to listen to him.

"I also would be interested in the Scandinavian model: all incomes are equally probable in a restricted interval, in an eco-window. Let me obtain that gintropy and Gini index, please!"

So Ken also gained the scene in front of the board. He presented the following calculation:

→ Eco-window gintropy, page 156

"As you realize the gintropy here also restricts to the interval of allowed incomes. However, its value is far from being a constant; it describes a piece of a parabola. The same is true with the gintropy function of the cumulative rich population. Its form, $\sigma = 3G\overline{C}(1-\overline{C})$ with G being the Gini index, reminds us to the formula of entanglement entropy."

"In particular, when the lower bound is zero, since then $G = 1/3$,' remarked Vilmano. 'Indeed then the gintropy is simply $\sigma = \overline{C} - \overline{C}^2$."

"What is entanglement entropy?" asked Ego.

Instead of Vilmano, Ken answered.

"In quantum physics people use the expression $\mathrm{Tr}(\hat{\rho} - \hat{\rho}^2)$, with $\hat{\rho}$ being the density matrix, or operator. That describes the statistics of quantum states. This entanglement entropy is zero for pure states, when the density matrix has only diagonal elements, with values either one or zero. Trivially in both cases all $(\rho - \rho^2)$ entries are zero. By the way this is a Tsallis-entropy with $q = 2$."

"But it is impossible that an economic model could have to do with quantum behaviour!' picked Ginny on Ken. 'The analogy is at most formal – and nothing more than that."

"Good enough," shrugged Ken.

"After reviewing these interesting cases and the mathematical definitions of describing economic inequality, let us think about the report we have to deliver until tonight," warned Rudolph.

Ego rose to speak.

"I totally agree with this last remark. And in order to facilitate this work, I suggest that you take part in an excursion to a virtual reality after lunch. That must help you to describe the results without mathematics."

"Sir, if this is possible at all, who will program this virtual reality? How shall we take part in that?" asked Ken.

Ego wrinkled his forehead.

"Did not I say? I can reprogram the machine in the cellar. According to your formulas it will broadcast a simulation based on your suggested cases of societies. The rest is a question of hardware – unknown to me."

The team became alert in a second. They were in high mood all over the lunch break, and afterwards they gathered at the staircase, leading down to the secret laboratory.

Virtual journeys

Ego appeared with determined steps. He wore a casual dress, fit to a real excursion.

“Please, follow me,” he said, and stepped forward to a corridor on the left, neglecting the surprised faces and "Oh"-s. This route led to somewhere else than the cellar.

“I did not know that we have this path here,” noted Santino.

“I have déjà vu, from my dream,” echoed Ginny.

The curvy path led to a dark and spacious chamber with several seats in about five rows, as in a private cinema. Ego closed the door behind the last person and went to the front of the audience.

“Please, get seated, ladies and gentlemen. This is the most modern virtual reality cinema chamber we could build. You find connection helmets and gloves, wearing those you shall feel as being there in the sceneries.”

“Shall we be able to interrupt the illusion? As an emergency exit?” asked Fred.

“Yes, there is an emergency protocol. It feels as you would die in the illusion, but then you will awake here. You have to think strongly on the phrase ’timit’. Yes, it is an anagram.”

“What is on the menu? Where shall we go first?” wanted Ken to know.

“I have programmed your journey to utopia, a human world with undetermined time and place. The cities you visit practice one of your model societies you have computed this morning.”

“It sounds like fun!‘ exclaimed Vilmano. ’Let us start it!”

It became dark in the cinema hall, and the seven scientists of the research team connected their helmets and gloves to the supercomputer which must have worked behind the scenes. Possibly it was the mysterious machine in the cellar, who knows...

Each of them lost the conscious connection to the cinema hall and merged into his or her individual dream. Nevertheless, unexpected meetings are not excluded...

– Fred’s illusion.

Fred found himself in his native country, Hungary, in the city of Budapest. At least it looked so at the first glance. It was just such a feeling, no details could be sensed sharply. He was walking in the Pest side, near to the Madách square, towards Astoria and the richest and most beautiful synagogue of Middle and

Eastern Europe in the Dohány street. It was a warm spring day, people happily walked around, no hasty tumult as in "normal" times.

The women, ladies and girls, matrons and mommies looked charming, well-dressed, with clean and cosmetically enhanced faces and haircuts. Everyone looked relaxed and happy, smiling, and at the same time not being really interested in the other walkers. Fred went on, turned towards the river Danube on Rákóczi street. Something strange occurred to him: no noise of car traffic was heard. Still, there was a huge flow of cars, following each other closely. They just did it in a deep silence. *“These must all be electric cars!”* thought Fred suddenly.

The shop windows were clean and huge, stuffed with pieces for trade. He randomly chose a smaller shop in the side street, and went in.

“Good afternoon, Sir!‘ the clerk welcomed him with a velvet smile. ’I hope you are having a nice day! How may I assist you?”

“Uhm, I just wanted to have a look.”

“Sure, we are here for that purpose. Please call me, if you need help.”

Fred was looking around. Neatly packed devices in dozens, about most of them he had no idea for what use they shall be. The clerk beckoned a young girl and directed his finger to Fred. The girl came closer.

“What would be your primary need, Sir?”

Fred was surprised by the deep alto tone. “Oh I will not be left alone here,” he thought. Loudly he told,

“I need some time to decide it. Could you inform me about the proper use of this device?” He pointed to a small black paper box on the closest shelf.

“Oh, this the latest cry is in perass! You hav’ good eye, sir!”

Fred looked perplexed.

“Excuse me! What exactly is a perass?”

The girl looked with a shadow of suspicion on her face. Is it going to be a joke on her account? But then she formed her trader face for esteemed customers.

“In the old language a ’secretary’: perass shortens the clumsy phrase ’personal assistant’. This is a highly customizable 5G-9G mobile net connection of type Y-fon. Everyone is crazy for that.”

Fred looked lost. He did not feel he would need that device, nor that he would have enough money with him. Was not it just an illusion, a dream? The girl’s voice interrupted his thoughts.

“If you buy two you get the second for half price. And we personalize it for you with no extra costs. You must want it!”

“Well, if you insist... why not? Let us make a deal.” Fred was curious about the process of "personalization".

“Very well. Please come to the desk and confirm with your chip.”

Now Fred was in trouble.

“I am afraid I do not have my chip with me. Perhaps I’ll return later...”

Both the girl and the clerk suddenly changed style. The air seemingly froze.

“Are you of class B? What are you looking for in this area?”

“I do not know. I apologize.” Fred run to the exit in panic. He managed to get into the street and get along some hundred meters without any interception. There two muscular men, in a kind of strange uniform, stopped him.

“Please, identify yourself!”

“I would, if I know how.” Fred could not escape. So he just stood and waited for the next step.

“Since you cannot certify your A-classness, we have to remove you. Please get in our car.”

A soundless car was stopping beside them, Fred was invited to the back seat. They accelerated and turned, and with a high speed they reached the outskirts of the city in ten minutes. Fred was released there and put on the street.

These surroundings looked different. Many, far too many people on the streets, poor looking and smelling, hastily and suspiciously looking around. No smile on the faces. Time to time some collided with Fred and went on, said nothing. No word of excuse, or any other unnecessary politeness.

The faces looked very mixed. White Europeans in the minority, most of them Asiatic, Chinese, Indian or Afro-Indian with all shades of color from light brown to the darkest black. The clothes looked colorful but randomly collected. And the noise was strong: honking, brake squeak, engine roar, shouts in various languages.

“Welcome in the world of class B. You are the fundament of the society. Hail to the workers!”

These words came from a public speaker device from above. Also cameras were seen everywhere on lamp stands and other places.

Fred broke his way to a device which reminded him to an ATM. It was an information device. He managed to touch the screen, although he feel remorse not having gloves on by this action. The system looked logical and was supported by icons. Not much text, almost no numbers. Do not overload class B people – Fred realized.

"So this should be the two class society?" he contemplated. Certainly, without difference between A and B, and their income a and b, there would be absolute equality, like in theoretical communism. *Is it good?* He could not decide.

Suddenly a group of adolescent boys appeared in punk clothes and neon-colored hairs. They looked like hunters, and he realized that he was discovered as the next prey. It was far from Fred's nature to begin with a street fight. He pushed the red button and said loud "timit, timit!" By that he felt as dying. Some long-long seconds later he realized he was sitting in the private cinema hall of Ego's castle.

– Vilmano and Ginny

Vilmano found himself sitting in a luxurious private jet and sipping Cinzano. He looked out of the window to see blue sky – from above it is always blue – and some scattered white clouds below. A friendly voice was turned on: *"Here is your captain speaking. We are approaching Great-London, and will be landing in 10 minutes. Please, close your electric devices, and secure your seat belts. We hope you had a pleasant journey, and we can welcome you on one of our flights soon again."*

Vilmano enjoyed the flight and the super soft landing. Indeed, no shaking was experienced, which – as he remembered – often occurs by landing against the wrongly directed wind. He did not have luggage other than a light suitcase with his laptop in it and a hat fitting to his beige summer suit. He walked through the reception part without interception, none asked for his pass or luggage.

As the automatic door at the end of the green corridor opened he suddenly found himself faced to a crowd of people. In the first row he spotted a nicely dressed young lady with mahogany red hair in a deep green skirt and white blouse set. She reminded him to Ginny. She was holding a sign: Dr Zotti.

Vilmano went to the young woman, and introduced himself:

"Vilmano Zotti at your service, lady!"

She smiled, "We were waiting for you, Dr Zotti. I am excited about your arrival. Please come with me, we are a little too late for your lecture."

What lecture? Vilmano panicked a little, but this emotion was not sharp, it died out in a soft, dreaming feeling of happiness and gratefulness for life.

They have chosen the underground, the fastest way to the city. Ginny talked vividly, sometimes she giggled like a teenage girl. She asked some polite questions about Vilmano's journey, and about the state of affairs in Florence and Italy, and a few questions about Vilmano's theory of gintropy.

They stopped near St Paul's, went to the surface, and climbed up and over the walking bridge towards the Globe Theatre. On that side they arrived at an unsuspicious looking business building full with glass and metal. They were lifted to the top floor, and Vilmano was pushed into a room with about a dozens of persons in business suits. Ginny smiled and took her seat at the front.

Vilmano felt like when the dream is sliding into a nightmare. *What do they want from me?* He filed his throat.

"I suppose you want to hear a lecture from me... about what? Gintropy?"

There was a middle-sized laughter, which is a good start.

"However, since you all know my work and read my papers, we can skip that part. Please ask your questions."

After a shocked silence people warmed up. The questions came.

"Dr Zotti, is it true that the Pareto percentage has some limits?"

"Can we have a chance to return to naturalness?"

"Would more equality ruin business?"

"What do you think about taxing all transactions?"

Vilmano was thinking, even in this dreamlike stage of him. Perhaps his subconscious was sending messages.

"The answer is 1000, one-zero-zero-zero, the binary eight. More precisely: yes, no, no and no."

Some laughter again.

"But I do not want to be so tight-lipped. The Pareto percentage has a natural lower bound by definition: when (1-p) percent of the population owns p percent of the wealth, say 80/20 means p=0.8, 50/50 would mean that the half owns the half of the wealth in total equality for all. What is more interesting, if given a distribution of the wealth, that function may limit the possible Pareto ratios."

Vilmano continued and walked up and down. This was his favourite teaching style.

"Since the Lorenz curve is always above the diagonal, the orthogonal to it cannot cut this curve for $p < 1/2$. The question is that whether such distributions exist which do not allow for coming close to 1/2."

"Let us take the exponential distribution of wealth. Then the Pareto ratio is fixed to 68/32, approximately. No other solution is possible, we have a single Lorenz curve and a single cut point with the Pareto requirement. For a Pareto distribution with fat tail, the characteristic point is even above this: only less people can have a bigger fraction of the total wealth. Usually 80/20 is quoted, but anything is possible, say 90/10 or 98/2 etc."

"It is a more interesting question whether we have a chance to restore an exponential distribution once we have established a fat tail. I opt for no. The overall inequality measure, the Gini index for the exponential distribution is 1/2, while for the Pareto distribution – we are experiencing in capitalism – it must be bigger. Contrary to claims found in the internet: certainly those conclusions must have been reached not by extrapolating the distribution, but just using raw data in a restricted window."

"More equality would not directly ruin business. In a sense it would heat the wish to increase inequality, at least if it can be realized, and the earned money can be spent for rational purposes afterwards. Or for irrational ones, which means a shorter process."

"Business in my perception has to do with the dynamics and forces behind the evolution of wealth distribution, not with its actual value or Gini index. Although the gintropy, the density of the Gini index between the Lorenz curve and the diagonal, shows remarkably analogue properties to entropy, positiveness, convexity, etc. – about its tendency to grow we know nothing. Economy does not look like a closed system; most businessmen do not behave like realizing that they throw stones in a glass house."

Vilmano was drinking from the glass on the table. He relaxed a bit.

"Finally taxation. It usually rescales the value of incomes, but not relative to the average. A flat rate does not change anything, as little as inflation. Re-shuffling is really a powerful tax system, usually cutting the high end edge of possible incomes. This may deform the distribution, but it will also be undergone by not absolutely legal means."

"On the other hand taxing speculative transactions could prevent the development of unstable bubble explosions on the financial market. As most people, I agree with it, just let me do my business for a last time before that."

Vilmano gathered again laughs. His style and lecture was liked in this circle. Ginny looked at him with haze in her eyes. *"Damn it!"* thought Vilmano, but it sounded like "timit". So suddenly he felt weak knees and darkness came upon him. He woke up in his chair in the cinema hall.

– Ken, Ludvick, Santino and Rudolph

Ken became aware of far voices, damped, as if several people would talk at the same time. He could not understand what they were talking about, some

German sounding fractions reached his ear, but certainly not English. The air felt warm, as in early summer. He opened his eyes.

He was sitting in a garden, below a huge castania, a chestnut tree, at a big wooden table. In front of him a big mug of fresh beer. He drunk and felt the taste, just a little bitter, otherwise cool and refreshing. Among the best beers he ever have.

He was in a *Biergarten* in South Germany, most probably Bavaria. At the same table he recognized known faces: Ludvick, Santino and Rudolph. He smiled after wiping the beer foam around his mouth.

"How did you manage to appear in my dream?" asked the three companions.

"I can ask the same from you," replied Ludvick.

"We four are connected? What a pity!" exclaimed Santino.

"Since it seems so, let us make the best out of it,' suggested Rudolph. 'If we cooperate, we may find out sooner in which simulation we are."

A local waitress appeared at their table, wearing a neat dirndl, the local traditional dress, rich in material and lace markings.

"Are you enjoying your time, gentlemen?"

"Yes, great! Thanks for asking!"

"Can I do something more for you?"

"Oh, in fact you can,' started Rudolph with a serious tone. 'We are strangers here, and would like to know more about your land."

"But before that,' interrupted Ken and smiled with a drunken grin to the waitress, 'I beg, bring us one more round of cool beer."

"Four mugs of beer, ordered," confirmed the maid and disappeared.

"We will not progress fast this way," shook Rudolph his head.

"Let the youngster,' intervened Santino, 'one more beer I could drink myself."

"But what if we get too sleepy in our dream?' played Ludvick the role of a philosopher. 'How many levels deep can we sink?"

"Forget it and enjoy!" suggested Ken. And he showed example by drinking again.

An ageless man approached their table and called:

"I have heard, you are strangers. I can describe you our beautiful country and explain our living style as none else. I work at the Statistics Office of this county."

Three of the circle at the table looked taken aback, but then Rudolph made an inviting gesture.

"Sure, why do not you join us for a chat, Mr... ?"

"Müller is my name. Johann Sebastian Müller at your service."

The members of this virtual expedition team also introduced themselves one after the other. Herr Müller was impressed.

"So you are scientists? Physics, informatics, mathematics? I suppose then I can talk to you in the language of numbers, *nicht war*?[2]"

"Please, do not overdo it,' warned Ken. 'We are in the middle of drinking the local beer, which tastes me enormously today."

The man smiled knowingly.

"Very well then. In what are you interested most?"

"We are really ignorant here. Please introduce us into your social and economic system first," promoted Santino the idea. He wanted to know in which simulation had they ended up.

"Hmm. Naturally, our system is the best of the world's. Carefully constructed mixture of liberty and limitations: the individuals are free, and still the society is well regulated."

"I hope not only by the brute forces of the free market," noted Ken.

"Not only. Our primary directive is to restrict wealth between limits, but within these limits let everyone to find his or her own fortune."

"Oh, this must be the eco-window simulation." thought Ludvick. On the faces of the others a similar recognition reflected. Rudolph nodded knowingly.

"What about the net income distribution?"

"Well, the frequency of having any income between the lowest and highest value, say a and b, is more or less uniform. This is cared for by the Taxation Council, who regularly announces the actual directives for the next month taxation rules."

"You update it monthly?" wondered Ludvick.

"Why not? In the era of quantum informatics this is not an issue any more."

[2] Right, is it not?

The local man waved to the waitress, showed his thumb up for one more beer and she nodded. She will bring five mugs next.

"And how do you fix these limits? What if the low value is near to zero?" wanted Santino to know.

The man shrugged, "There is a basic income for living. Otherwise we would spend too much for supporting the socially weak. It is smarter to avoid instability at this end. In our constitution we agreed on one tenth of the average as the lower limit."

"One tenth? Why not one third or one twentieth?" intervened Ludvick.

"You are asking too many questions. It just happened so. Since then we have peace and well-being. At least we, the real citizens."

Rudolph started to feel some unspecified danger. He tried to smooth the emotions.

"And what about the upper limit? What principle guides you by fixing that?"

The man calmed somewhat. He wrinkled his forehead.

"Well, less is publicly known about that. There was an agreement among the really rich in a series of private meetings. They agreed finally on a level which will not harm them more than a possible social turmoil of the poor."

"Nice, so what is the value?" asked Ludvick.

"Mathematically it does not matter, as long as it is much, much greater than the lower limit. That is common wisdom here and taught in the schools."

"But, more closely: it is ten times of the average or even more? It is not unjust?"

The man made a wrinkled face.

"Who should then decide that value in your opinion? The strangers, the immigrants, or the outlaws?"

"Perhaps all together..." tried to implement Ludvick his enthusiastic idea of human equality. But it had a terrifying effect.

The man's face became dark red, he jumped up from the table and called something in his local dialect. Then other men appeared and started closing by. They wore brown shirts and short trousers and some signs on their clothes.

The four of our seven scientists became alerted. They were petrified for two seconds, and then called "timit" concurrently. They died immediately – the feeling of that was real. They waked up in the small cinema hall and looked disturbed.

Scientists' report

After all had their virtual dreams and experiences, Ego decided to collect them on the terrace, served with coffee and pastries. The afternoon sun was a real pleasure. Then he stood in the middle of the small tables and announced:

"Professors, it is your last chance to write your report about inequality and its manipulation tools. Please gather in the lecture room and work!"

The society did so and sat in the lecture room in a quarter and an hour. They discussed in smaller groups, and then all seven together, frequently using tablets and projecting text passages. At around 7 pm Rudolph handed their report to Ego over. All felt exhausted, and returned to their rooms for a refreshment.

Ego took the file of the report, went to his private study room to read it undisturbed. Finally! He sipped on his whiskey a little, and then let the text roll on his private screen. It looked like a typical report written by scientists.

REPORT ON INEQUALITY

Its measure and its role in complexity

RUDOLPH GENIUS, SANTINO WALLIS, VILMANO ZOTTI, KEN KENSON, VIRGINIA ASTRA, LUDVICK HOLZMANN, FRED SERENYI

Preliminary mathematical results and suggestions for further research

Ego had a sigh. It is going to be a scientific paper, not what he can sell to the sponsors. Too bad. But he continued to read for the time being.

> Before asking for means and controls to influence inequality, in particular economic and social inequality, we have to agree on its definition.
>
> Any rational, scientifically approachable definition has to include a measurement prescription, paired with a quantitative output, in order to serve as a basis and test the results of the mathematical model in the background. We, the Authors, shall not spell out mathematical formulas in this report, but we give this warning in order to put the following statements into a proper perspective.

"What the heck!' thought Ego. 'They start with pushing away their responsibility. Let us see what are the results of this exhausting week."

> Income and wealth inequalities can be measured by statistics, mostly carried out by national statistics offices or taxation authorities. For the market economy the net income distribution, the income left in

> private hands after taxation, while for the social stability the amount of collected tax is important. So in principle before and after taxation the distributions differ, the question is how.

Indeed, the question is how. And which is the best way to achieve the ideal distribution.

> Certainly after taxation all incomes are less than before, unless we allow for "negative taxes", i.e. take social help money also as income into account. In this more general setting income can be shuffled from the rich end to the poor end.
>
> Some other factors on the other hand may shuffle income from the modest poor end towards the riches: e.g. inflation, a steeper price increase for articles characteristically sought for satisfying basic needs than for luxury needs. What is basic and what is luxury, of course is loosely defined, and depends on the common sense in the given society. It also changes with time and achievements in the common good.
>
> The income distribution is usually inspected over positive values, a natural first approximation to wealth and income. The inequality inherent in a distribution is attacked by comparing two cumulative measures: the wealth bigger than a value is plotted as a function of the population fraction owning these goods. This is the classical Lorenz curve.

"Aha, here starts the science voodoo jargon," noted Ego.

> The Lorenz curve always runs above the diagonal, for positive incomes. Its area closed with the diagonal is the half Gini index; a single number characterizing the overall inequality between rich and poor. Its value is always between zero and one, or hundred percent.
>
> Commonly known experimental distributions are close to the Pareto distribution, showing a fat tail. The fat tail means that the rich are more abundant than they would be in the case of a more natural distribution. "Natural" in this context means a distribution suggested by the energy distribution among atoms in thermal equilibrium.
>
> Studies of the mathematical properties of this inequality measure reveal that the density of the Gini index in the area between the Lorenz curve and the diagonal behaves like entropy, a general measure of randomness, complexity and chaos. In this sense the analogy is tempting. However, we do not know about its spontaneous tendency of growing since economy and society are not closed systems. They live in interaction with the environment – a bigger, although far from being an infinite system. Ecology may strike back if the greed of

> increasing capitals govern human actions onwards in the future.

"Oh, it reads like a political pamphlet,' ironized Ego. 'What shall I do? They really left out formulas, but they are not telling how to fix the problems."

> We have compared different models from the viewpoint of the Gini index, the integrated inequality measure. The trivial model is extreme communism, when everyone possesses the same income or wealth. Here the Gini index is zero and not much is left to discuss. Unfortunately, the dynamics and manipulation possibilities also cannot be studied in this extreme case.
>
> Our next model for theoretical study was a somewhat more realistic version of it: two classes with two fixed incomes constitute that society. Then one obtains a nonzero Gini index, it depends on the incomes and on the fraction factor between the class A and class B people. In case of huge differences between the two classes, when the income in the higher class is much bigger than that in the lower one, the Gini index depends solely on the fraction of riches versus the poor. In fact, it equals to the poor fraction, therefore it vanishes if everyone belongs to the upper class, and it is maximal if (almost) everyone belongs to the lower class. For the same income in the two classes we are of course back to the communism with zero Gini index.

"The most trivial part is the longest!" complained Ego half loud. Then he realized it and bit his lips in anger. He read further.

> We also examined another model, when the incomes have uniform frequency distribution between two fixed values, a and b. Then the Gini index is indeed bearably low, for much higher b than a it approaches one third. This is a value with which modern societies would be consent. Nevertheless, if the difference is smaller, the Gini index is less than one third, it may shrink even to zero when the difference vanishes. For a typical value of one fourth (twenty-five percent), the ratio of high to low edge income is seven to one. For a Gini index of one fifth (twenty percent) this ratio is four to one.

"Not exactly a Scandinavian social democracy,' commented Ego silently. 'But something like an artificial restriction leaving some freedom between two borders. Interesting!"

> Finally, we examined a model we consider as most realistic. In this the income distribution is continuous over the half-infinite interval of positive values. The form of the distribution is that of the classical one, suggested by Pareto; it shows a power-law like decreasing tail, governed by one parameter. The corresponding Gini index is always bigger than fifty percent, in the borderline case of the "natural"

> (exponential) distribution it is exactly fifty percent. Here the bonmot of an 80/20 society, where twenty percent of the population owns eighty percent of the total wealth, is fulfilled at a given value of the only parameter in the general scaling distribution. Such a statement cannot be made below a critical value: 68/32 is the least. Certainly it can be larger than 80/20, up to 100/0.

"Are they describing our real society?' wondered Ego at this passage. 'Are not we closer to 90/10?"

"So what! Nothing about the possibility how to reach $G = 0$." Ego felt cheated.

> To the question which is the best Gini index or the best distribution, that would make a society superior to others, we cannot give a scientific answer. For that a number of further questions have to be investigated: What is the dynamics of gintropy? Does the usual market behaviour prefer one or another distribution? Can it be basically changed what we have now or not?
>
> It is possible that some further research may attack these questions, given more time, more effort, more researchers and more money.

At this end Ego was deeply disappointed. He shook his head.

"I cannot give this to the sponsors! I shall have to write my own report, and now."

Ego's report

How to eliminate inequality: the gintropy project

EGO ZEUS GEORGOS

So far it was easy. But what to write? A long version? Or a short one? Ego decided for the short briefing.

> The inequality cannot be eliminated. That would lead to communism, killing all dynamics. That will be unstable against splitting to a two class society.
>
> The present situation with the global spread of capitalistic economies, even if augmented with more or less social societies, forces an inequality upon us which can only be over the natural minimum. There is more contest than necessary, exactly this factor dynamises.

> The gintropy project, collecting researchers best in their genres, mathematics, informatics, thermodynamics, did not deliver a solution for the elimination of inequality. Their results even question, that whether this elimination would work or would it do any good.
>
> The closest to a clue for making societies more just and people more content with life and rules is given by an interesting example of academic interest: the eco-window model, supposing a uniform chance of any income between fixed boundaries. Here the upper boundary can be even extremely large, that still makes less inequality than the present capitalism.
>
> The only problem with this model is that it cannot be realized. Having the same uniform chance for any income in a forgiven window sounds very artificial. Humans want an income proportional or at least growing with their growing work or money invested. Therefore, this window-model is impossible to be introduced.

Ego sighed. The result of all the effort is at the end negative. Fully unsatisfactory. This may cost his comfortable position.

Having thought about this Ms Kunz entered into the private study room.

"Boss, I would like to talk to you. It is important."

"Only because it is you, Ms Kunz. What is it?"

"I quit."

"Why, so sudden? Did I treat you wrong?"

"No, not that. I am just fed up. Tired. Want something else."

"I am relieved, if it is not my fault." While saying that, Ego felt an increasing darkness around his heart. A cold finger touched his soul.

"It is not your fault, but my chance for better."

Having told all that, Ms Kunz, Emma, left Ego, not looking back. His problems were no more hers. She felt relieved, and went to fetch Ludvick.

Ego was left behind. His disappointment reached a new high. *"I have to straighten it out somehow. But how?"* was his first reaction. *"After such a fiasco in all my projects I shall be killed."* But he loved life too much. He was not a suicidal type. He made a different plan.

Ego had a secret project. He knew that the machine in the cellar can be used for time travel; jumps into the past in particular.

Yes, it is prohibited by moral and good taste. Yes, it may cause paradox situations with devastating effects. At least in one of the parallel universes.

He did not see any other way out. This project must be successful, not like it looks by now. He decided to risk his travel back with one week.

Taken the key device from his secret drawer, Ego Zeus marched to the cellar. It was easy to block the door behind him, the others anyway were on their ways to leave. *"Like rats on the sinking ship!"* thought Ego bitterly.

Closing the lab's door, he comfortably arranged himself in the main seat. He attached the personal coding device, and programmed the date exactly one week backwards. *"I can repeat this as many times as necessary."* This idea satisfied him.

Pushing the initiating sequence the tiny silver cables connected to his neck. He felt some dizziness, and then falling into a dark and cold space. A slight feeling of an alarm disturbed his last memory.

The time paradox lasted heavily on him. Ego decided to break out from the seat, he ripped out the cables and lost consciousness. He fell out from the cabin and stayed lying on the floor.

That was the position he was found there on the first night of the project. As an unrecognizable stranger, an unknown man.

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Chapter 8

Formal background

In the forthcoming pages the inserts of mathematical formulas and derivations follow. To each of them there is a jump, hooked in the text of the story, and then a jump back to continue the main reading. A few sentences introduce each insert, relating it to the main actions in the plot of the story.

However, it is strongly advised *not* to read these inserts one after the other, but rather to follow the main story and discussions, jumping to and back. Otherwise it will look as an unmotivated, chaotic salad of various constituents.

Only geniuses or people living with autism might enjoy this.

On the other hand, it has been tried to improve this salad by seasonal spices: short descriptions from where in the text and why are these formulas and their derivations. The Author earnestly hopes that this helps.

The first mathematical insertion is about permutation entropy, as a preparation by Ludvick to his university lecture. This must be known by experts from several textbooks; here it is repeated for the sake of the feeling, how Boltzmann's entropy formula can be rooted in the counting of permutations of equivalent microstates. Later there will be discussions about other entropy formulas, too.

permutation entropy from page 8

Let the number of permutations of N microscopic possibilities be $N!$, meaning all are equally probable. Now let us examine the relative occurence frequency of more restricted states in this pattern, having in order $N_1, N_2, \ldots, N_W$ realizations. Then the degeneracy number of this case is equal to that of the repeated permutations:

$$\mathcal{N} = \frac{N!}{N_1!N_2!\ldots N_W!}. \tag{8.1}$$

The corresponding entropy,

$$S/k = \log \mathcal{N} = \log N! - \sum_{i=1}^{W} \log(N_i!), \tag{8.2}$$

will be a difference in logarithms. Since in our world we usually deal with very many atoms or molecules, i.e. with a huge number of microstates and definite, finite number of macrostates, we have $W \ll N$. Taking into account all possibilities, clearly the sum of all cases is equal to the total number of realizations before making distinctions by the macrostates:

$$N = \sum_{i=1}^{W} N_i. \tag{8.3}$$

We need to estimate the logarithm of factorials for big numbers. That is done by the Stirling formula, from which the two leading terms suffice here:

$$\log N! = N \log N - N + \mathcal{O}(\log N). \tag{8.4}$$

Applying this to the above, we arrive at

$$S/k = N \log N - \sum_{i=1}^{W} N_i \log N_i. \tag{8.5}$$

Measuring the probability of the occurrence of the i-th macrostate by the fraction $p_i = N_i/N$, we arrive at

$$S/(kN) = \log N \left(1 - \sum_{i=1}^{W} p_i\right) - \sum_{i=1}^{W} p_i \log p_i. \tag{8.6}$$

It is obvious that the leading term for large N provides the normalization condition, leaving us with a finite entropy formula

$$S/k_B \; = \; -\sum_{i=1}^{W} p_i \log p_i. \tag{8.7}$$

Here $k_B = kN$ is the measureable constant, indeed in everyday physical units a large value. N must be in the order of the number of atoms, the Avogadro number.

Back to page 8 →

Another protagonist in the story, Ken, is preparing his lecture about the measure of information and its connection to the length of bit strings. This is also important background knowledge in recognizing the central role that entropy plays in statistical physics and therefore also in modelling complexity.

information measure from page 11

How to measure information? In the minimal length of a string of bits describing the content. A bit can have two values, 0 or 1, so an N-bit long string can be chosen from 2^N configurations.

The fastest way to find a given one from all of such strings is to ask N yes/no questions, by each answer exactly halving. Then we need $S = \log_2 N$ steps. Interesting enough, this coincides with Boltzmann's entropy formula for a uniform probability $p_i = 1/N$

$$H \; = \; -\sum_{i=1}^{N} \frac{1}{N} \log_2 \frac{1}{N} \; = \; \log_2 N. \tag{8.8}$$

Perhaps more is behind. Claude Shannon gave a proof that the shortest length of a coded message, a bitstring, from which the decoding can be made perfectly, tends to be near to this quantity for large N:

With X a random variable, with value distribution $p(x)$, the length of the coded word $\ell = |c(x)|$ for the optimal coding $c(x)$ satisfies

$$H \; \leq \; \langle \ell \rangle \; \leq \; H + 1. \tag{8.9}$$

Here the expectation value is built with the distribution $p(x)$.

Note: for non-binary codes (an alphabet with a letters instead of two), the quantity $H(x)$ is to be divided by $\log_2 a$. It is equivalent with the use of $\log_a$ in the entropy formula.

Back to page 11 →

When generalizing the concept of entropy it is of primary importance to be aware of the basic properties, axioms, defining the classical entropy – probability formula. In particular its convexity, the definite sign of its second derivatives with respect to physical parameters, is unique: This is also the basis for designating an arrow to time, as in Boltzmann's H-theorem or in the uncountably many times cited second law of thermodynamics.

Ginny formulates this convexity property for a generalized, so called trace formula, for the entropy, while being bored in a seminar talk in London. The definite sign of the second derivative has consequences on possible expectation values weighted by positive factors between zero and one. Exactly how probability is defined.

convexity of entropy from page 12

A typical entropy formula has a single maximum in terms of the most relevant parameters. If it possesses a negative second derivative over the possible interval of its independent variable, i.e. $f(x)$ is such that $f''(x) < 0$, then it is easy to realize that any convex linear combination, meaning a weighted sum with weights between zero and one, any convex combination of x_i values and $y_i = f(x_i)$ values, as coordinates lie in a convex polygon under the graph curve of $y = f(x)$. Consequently from its y-coordinate it follows

$$\sum_i p_i f(x_i) \;\leq\; f\left(\sum_i p_i x_i\right). \tag{8.10}$$

Here all $p_i \in [0,1]$ and $\sum_i p_i = 1$ must be fixed, in order to move inside the area stretched by the corner points of the polygon only.

The classical entropy formula uses $f(x) = \log(x)$. For choosing $x_i = 1/p_i$ as a special arrangement, we prove that such formulas are maximal for the uniform distribution:

$$S[p_1,\ldots,p_N] \;=\; \sum_{i=1}^{N} p_i f\left(\frac{1}{p_i}\right) \;\leq\; f\left(\sum_{i=1}^{N} p_i \frac{1}{p_i}\right) \;=\; f(N) \;=\; S[\frac{1}{N},\ldots,\frac{1}{N}]. \tag{8.11}$$

On the other hand choosing $x_i = q_i/p_i$ with $\sum_i q_i = 1$, i.e. the q_i weights sampled from another probability distribution, we obtain the Gibbs' inequality:

$$KL[q;p] \;=\; \sum_{i=1}^{N} p_i f\left(\frac{q_i}{p_i}\right) \;\leq\; f\left(\sum_{i=1}^{N} p_i \frac{q_i}{p_i}\right) \;=\; f(1). \tag{8.12}$$

For the usual entropy formulas $f(1) = 0$. Applying the logarithm function

$$S[p] \;=\; -\sum_{i=1}^{N} p_i \log p_i \;\leq\; -\sum_{i=1}^{N} p_i \log q_i. \tag{8.13}$$

Back to page 12 →

A straightforward generalization of the logarithmic entropy formula due to Boltzmann utilizes a generalization of the logarithm function. The same way additivity at the factorization of probabilities is generalized as a group operation beyond addition. But since group operations are by definition associative, to any group entropy a formal logarithm, an additive combination can be constructed.

The lecturer in the seminar was talking about how to construct this formal logarithm, how to relate the function of entropy, $K(S)$, used instead of S, guaranteed additive, while the original entropy was not.

formal logarithm from page 12

The logarithm function in the classical formula is applied because of its special property of mapping a product to a sum

$$\log(xy) \;=\; \log(x) + \log(y). \tag{8.14}$$

But we can use any other function, $f(x)$, as long as it satisfies some basic requirements. Such as $f''(x) < 0$.

Can we have more restrictions? Associativity is an important step towards describing the composition of big, complex systems:

$$h(x, h(y,z)) \;=\; h(h(x,y), z). \tag{8.15}$$

This means that the composition $x \oplus y = h(x,y)$ describes a structure. Adding the requirement, that when composing with nothing, nothing will change, $h(x,0) = x$.

For such composition rules there always exists a monotonic and hence invertible function, $K(x)$, satisfying

$$K(h(x,y)) \;=\; K(x) \,+\, K(y). \tag{8.16}$$

This generalized logarithm of the formal group can be obtained from the composition rule, if it is smooth enough. Taking the y-derivative of the above at $y = 0$ delivers

$$K'(h(x,0)) \cdot \left.\frac{\partial h}{\partial y}\right|_{y=0} \;=\; K'(0). \tag{8.17}$$

Fixing $K'(0) = 1$ for suiting $K(x) \approx x$ at small values, we obtain the solution

$$K(x) \;=\; \int_0^x \frac{dt}{\frac{\partial h}{\partial y}(t,0)}. \tag{8.18}$$

Back to page 12 →

Well known generalizations of the Boltzmann entropy are the Rényi and Tsallis formulas. They are particular formulas, the first designed for being additive for a product formula, the second for being an expectation value. Santino Wallis' deliberations while preparing calculations for his forthcoming paper deal with these two formulas.

Rényi, Tsallis, from page 16

The classical entropy formula makes the entropy additive in cases when the joint probability of two independent events factorizes: $p_{ij} = p_i q_j$ with both p_i and q_j normalized to one. Then these are the marginal probabilities: irrespective of the state of the other component, the selected component is in a given state,

$$p_i = \sum_j p_{ij} = p_i \cdot \sum_j q_j, \quad \text{and} \quad q_j = \sum_i p_{ij} = q_j \cdot \sum_i p_i. \tag{8.19}$$

For making a sum from a product is always the logarithm in use. But the core function in the entropy formula is not necessarily the logarithm. It may be another factorizing expression, and then one takes the logarithm of the result. Exactly this lies at the heart of the Rényi entropy. Since $(xy)^q = x^q \cdot y^q$, the q-th power factorizes. Define

$$H_q[p_i] = \frac{1}{1-q} \ln \left(\sum_i p_i^q \right), \tag{8.20}$$

then the above test goes through with additive H_q. The property

$$\sum_{ij} p_{ij}^q = \sum_i p_i^q \cdot \sum_j q_j^q, \tag{8.21}$$

makes its logarithm, the Rényi entropy additive. The forefactor, $1/(1-q)$ is needed to have Boltzmann's formula in the limit $q \to 1$.

Very well. But Boltzmann's formula looked like an expectation value, while this generalized Rényi entropy not. Can it be healed?

Consider that whenever H_q is maximal, also its positive function is maximal. Check this:

$$S_q = \frac{\mathrm{e}^{(1-q)H_q} - 1}{1-q} = \sum_i \frac{p_i^q - p_i}{1-q} = \left\langle \frac{p_i^{q-1} - 1}{1-q} \right\rangle. \tag{8.22}$$

This Tsallis entropy is also maximal and it is an expectation value of something. This quantity can be called a "*deformed logarithm*". Either $S_q = \langle -\ln_q p_i \rangle$ or $S_q = \langle \ln_q(1/p_i) \rangle$.

Back to page 16 →

The Tsallis entropy formula, also called q-entropy, is usually presented as containing a generalized logarithm. However, there is an ambiguity depending on whether the minus of logarithm function of the probability or the logarithm of one over the probability is renamed. After the generalization these two ways are no more equivalent. Santino is deepened in this derivation at the beach in Rio.

q-log ambiguity, from page 16

The question is whether we generalize the logarithm function in the classical formula or the expression of which the entropy its expectation value shall be. Either

$$\frac{p_i^{q-1} - 1}{1-q} \; = \; -\ln_q p_i \tag{8.23}$$

or

$$\frac{p_i^{q-1} - 1}{1-q} \; = \; \mathrm{Ln_q} \frac{1}{p_i}. \tag{8.24}$$

For the classical natural logarithm function, recovered at $q = 1$, these two definitions coincide. For general parametrizations they differ. While classically $\ln(1/x) = -\ln(x)$, here – using the first definition – we have

$$\ln_q \frac{1}{x} \; = \; \frac{x^{1-q} - 1}{q-1} \; = \; -\frac{x^{1-q} - 1}{1-q} \; = \; -\frac{x^{(2-q)-1} - 1}{(2-q)-1} \; = \; -\ln_{2-q} x. \tag{8.25}$$

A further unusual property is reflected in the q-logarithm of the product. Noting that

$$x^{q-1} \; = \; 1 + (q-1)\ln_q(x) \tag{8.26}$$

we derive from $(xy)^{q-1} = x^{q-1} \cdot y^{q-1}$ that

$$1 + (q-1)\ln_q(xy) \; = \; [1 + (q-1)\ln_q(x)] \cdot [1 + (q-1)\ln_q(y)]\,. \tag{8.27}$$

This unveils the following composition rule:

$$A \oplus B \; = \; h(A,B) \; = \; A + B + (q-1)\,A\,B. \tag{8.28}$$

with $A = \ln_q(x)$, $B = \ln_q(y)$ and $A \oplus B = \ln_q(xy)$.

Back to page 16 →

One of the most famous formula in quantitative finance and econophysics is Pareto's rule, classically cited as the 80/20 rule. This is a compact formulation directing the attention to a seeming injustice in the wealth distribution. In the note below the definitions of the cumulative populations of rich and poor fractions and that of the wealth possessed by them are given. Vilmano contemplates on this while receiving an American coffee mug in Florence with the 80/20 sign on it.

Pareto's 80/20 rule, from page 20

Let us discuss a continuous model. The number of earners having an income between $y - dy/2$ and $y + dy/2$ be $N(y)dy$. For small enough intervals, $dy \to 0$, the proportionality with dy holds. The income distribution is described by the probability density function, PDF as

$$P(y) \; = \; \frac{N(y)}{N_{\text{tot}}}, \qquad \text{with} \qquad N_{\text{tot}} \; = \; \int_0^\infty N(y)\, dy. \tag{8.29}$$

The wealth fraction received by a population fraction are given by Pareto's law:

$$\int_{x_P}^\infty P(y)\, dy \; = \; 0.20 \qquad \text{while} \qquad \int_{x_P}^\infty y\, P(y)\, dy \; = \; 0.80\, \langle x \rangle\, . \tag{8.30}$$

Here x_P is the splitting income value; 20 % of the population earns more than that, and integrated this makes out of 80 % of the total earnings.

The PDF is normalized and defines the average income, too:

$$\int_0^\infty P(y)\, dy \; = \; 1, \qquad \text{and} \qquad \int_0^\infty y\, P(y)\, dy \; = \; \langle x \rangle\, . \tag{8.31}$$

Back to page 20 →

Beyond the 80/20 bonmot there is a whole distribution, called Pareto distribution, with fat, power-like decay tail for the very rich. Vilmano realizes in his notes that in order to arrive at the 80/20 rule one needs such a probability density function for wealth, that renders all people having more than 87 percent of the average to the rich fraction. This fact is surprising. The upper 20 percent might not be really rich.

Pareto distribution, from page 21

Several even if not all income distributions may have the required properties. Pareto found a certain functional form. The most important ingredient was its behaviour for large arguments, the probability for the really wealthy people. Of course, as much as one can know about their fortune.

The Pareto distribution is normed, and it is a function of the ratio of the actual income to the average income. However, one parameter is left over, the very power α in the formula

$$P(x) \sim x^{-\alpha} \tag{8.32}$$

established for large x-s. Satisfying the norm constraint one writes

$$P(x) = AB\,(1+Ax)^{-B-1}. \tag{8.33}$$

The expectation value can be related to A:

$$\langle x \rangle = \frac{1}{A(B-1)}. \tag{8.34}$$

The remaining unknown, B, can be obtained from the $80:20$ rule.

The general cumulative integral describes the population fraction having more than x, similarly there is a fraction of the wealth possessed by them:

$$\overline{C}(x) = \int_x^\infty P(y)dy = (1+Ax)^{-B} \tag{8.35}$$

and

$$\overline{F}(x) = \frac{1}{\langle x \rangle}\int_x^\infty yP(y)dy = (1+Ax)^{-B}\,(1+ABx). \tag{8.36}$$

Pareto says $\overline{C}(x_P) = 0.20$ and $\overline{F}(x_P) = 0.80$. Using his distribution the ratio of $\overline{F}$ to $\overline{C}$ at the splitting income x_P becomes

$$\frac{\overline{F}(x_P)}{\overline{C}(x_P)} = 1 + ABx_P = \frac{80}{20} = 4. \tag{8.37}$$

From here

$$\frac{x_P}{\langle x \rangle} = 3(1 - 1/B). \tag{8.38}$$

Then B can be obtained from $\overline{C}(x_P) = \left(1 + \frac{3}{B}\right)^{-B} = 0.20$. It is approximately $B \approx 1.41$ and $x_P \approx 0.87\,\langle x \rangle$.

Back to page 21 →

This inlet is about the surprising fact that increasing and decreasing something with the same percent results in a loss. This is also an everyday example of the non-additive behaviour with $q-1 = 1/100$, zoomed on in Ludvick's presentation at the first evening of the project. This character always tries to broadcast some deep meaning by simple means.

non-additive percents, from page 28

Increasing something with p_1 percent makes a multiplicative factor

$$F_1 = 1 + \frac{p_1}{100}. \tag{8.39}$$

Say for an increase with 20% the factor is 1.20. Let us look for the repetition of two such steps after each other, and set it equal to a one-step factor:

$$\left(1 + \frac{p_{12}}{100}\right) = \left(1 + \frac{p_1}{100}\right) \cdot \left(1 + \frac{p_2}{100}\right). \tag{8.40}$$

It is simple elementary school arithmetic to express from this the composite percentage:

$$p_{12} = p_1 + p_2 + \frac{1}{100} p_1 \cdot p_2. \tag{8.41}$$

This is one of the most common non-additive composition rules, with $q = 1.01$. cf. eq.(8.28).

Now let us zoom on the original tricky question. This is the case when $p_1 + p_2 = 0$. This gives

$$p_{12} = -\frac{p_1^2}{100} = -\frac{p_2^2}{100}. \tag{8.42}$$

Back to page 28 →

Fred presents his civilization model, as an example of a mathematical framework for complex systems beyond physics. Concentrating on a demographic pointer, the total population, a very simple dynamical model is presented in several steps. In the first step below the approach by the Clube of Rome in 1960 is reconstructed.

civilization mean field, from page 33

We study a mean field model, one civilization in an external environment. First question is how the population develops. I start with factors increasing it and then look for decreasing effects. The time derivative depends on the value quadratically, since – usually – two persons are needed to "make" a new one:

$$\dot{p} = \lambda p^2. \tag{8.43}$$

This dynamics has a solution, diverging in a finite time:

$$p(t) = \frac{p_0}{1 - \lambda p_0 t}. \tag{8.44}$$

The divergence time, $t_{\text{div}} = 1/(\lambda p_0)$ is the shorter the larger the initial population is. This time remains until the population explosion, as the Club of Rome conjectured it.

Back to page 33 →

Going on with his lecture Fred clarifies the exponential growth already suggested by Malthus in the 18th century. The main actors in his model consisting of differential equations are introduced gradually.

Malthus, from page 33

Reducing factors are natural or "unnatural" deaths, which I assume as a contant rate to begin with. Then our equation changes to

$$\dot{p} \;=\; \lambda\, p^2 \;-\; \mu\, p \;=\; \lambda\, p\,(p - p_M), \tag{8.45}$$

with $p_M = \mu/\lambda$, I call *Malthus point*. Then the solution modifies to

$$p(t) \;=\; \frac{p_M p_0 \mathrm{e}^{-\mu t}}{(p_M - p_0) + p_0 \mathrm{e}^{-\mu t}}. \tag{8.46}$$

For $p_0 < p_M$ this civilization dies out, for $p_0 > p_M$ it explodes as before. The time to the divergence is, however, now, longer, it occurs in a logarithmic scale $t_{\text{div}} \;=\; -(1/\mu)\, \ln(1 - p_M/p_0)$. The previous result emerges for $p_0 \gg p_M$, equivalent to $\lambda \gg \mu$.

Back to page 33 →

An important new element is to allow for a saturating population. This takes the possibility of finite resources into account.

saturated demography, from page 34

So we zoom on the factor λp^2. The larger the area, A, from which a civilization acquires resources the larger this rate is. The proportionality factor I call *resource efficiency* and note by r. On the other hand I reduce this rate proportional to the population number itself:

$$\lambda p^2 \;\to\; (rA - cp)p. \tag{8.47}$$

This effects a high saturation value in p/A instead of the divergence. Whenever $p/A > r/c$ the population will be reduced, below it it will grow:

$$\dot{p} \;=\; c\, p\,(p_S - p), \tag{8.48}$$

with $p_S = rA/c$. And yes, a dynamics of the area controlled by a given civilization also has its own dynamic law. And it is coupled to the population number, p, cannot be treated as an independent constant.

Back to page 34 →

Finally, to arrive at a realistic model Fred presents another equation describing the dynamics of the resource area. By doing so two differential equations couple and show various characteristic solutions for the development of demography. These formulas underline the successful presentation of data in graphs.

resource dynamics, from page 34

The area also follows a simple dynamics. It may increase and decrease. My main factor of increase is the population density, if p/A is above a threshold m, then new areas must be conquered. And the expansion of a civilization is concentrated to its frontier, which – expanding in two dimensions – goes like $\sqrt{A}$. In summary we solve the following coupled equations

$$\dot{p} = rAp - cp^2 - \mu p,$$

$$\dot{A} = \kappa \, (p/A - m) \, \sqrt{A}. \tag{8.49}$$

For $p < mA$ the influence of a given civilization shrinks and it looses resource area. In the equilibrium point, when the area does not change, $A = p/m$. Then the effective population dynamics becomes similar to the one we started with. As long as there are enough virgin lands to be annected.

Now the decisive factor is r, for $r > mc$ the population grows, in the opposite case, $r < mc$, the civilization disappears from the pool. But even for $r > mc$, there are two different phases of the history. As long as the resource collection area can be increased, at lower populations in the early history, $A \propto p$ holds and we have growth. In the late history, A reaches its maximum and p goes into saturation.

Back to page 34 →

Entropy is frequently considered as missing information. More precisely the mutual entropy and the mutual information can be identified with each other, up to a sign. Since the overall inequality in income and wealth looks like a complexity measure, its relation to entropy and information will be discussed later in this story.

The length of the optimally coded text is the base of the entropy-like treatment of information. The following derivation reveals how naturally the logarithm function arises by this procedure.

text code length, from page 38

Suppose we have an alphabet consisting of two letters, A and B. In a long text or picture file A occurs in 90 % of the cases and B in 10 %. If we code one by one we cannot spare anything on the length of a message.

However, we may code the doubles according to their occurrence frequency, e.g. AA by 1, AB by 10, BA by 100 and BB by 1000. These are 1, 2, 4 and 8 in a binary representation. The code lengthes are in order 1, 2, 3 and 4 bits. In a random text the average length per symbol by coding these diphthongs is

$$\frac{1}{2}\langle \ell_2 \rangle \;=\; \frac{1}{2}(0.81\cdot 1 + 0.09\cdot 2 + 0.09\cdot 3 + 0.01\cdot 4) \;=\; 0.65. \tag{8.50}$$

This means a compression onto 65 %, relative to the one-bit code length

$$\langle \ell_1 \rangle \;=\; 0.9\cdot 1 + 0.1\cdot 1 \;=\; 1. \tag{8.51}$$

By the use of the same coding for the quadruples, from $AAAA$ to $BBBB$, if their occurrence probability is a product of the previous diphthong probabilities, the result, $\frac{1}{4}\langle \ell_4 \rangle = 0.65$, is the same.

The bit-length of a binary number, x, is given as

$$\ell(x) \;=\; 1 \;+\; \text{floor}(\log_2 x). \tag{8.52}$$

Back to page 38 →

Following Ginny, who has showed how the logarithm function occurs in optimal length coding, Ken spelled out the Shannon formula, reminding to Boltzmann's entropy – probability relation.

codelength entropy, from page 38

Longer bit-strings, say of four letters, are coded by higher numbers, x_i. Say $BBBB$ by $10001000_2 = 136$, etc. The best to code the least probable one by the longest code, representing the biggest number, therefore $x_i = 1/p_i$ is an optimal choice. That also keeps the average, $\langle x \rangle = \sum_i p_i x_i$ constant.

The average code length becomes

$$\langle \ell \rangle \;=\; \sum_i p_i \ell(1/p_i) \;=\; 1 \;+\; \sum_i p_i \log_2(1/p_i), \tag{8.53}$$

by using Ginny's formula (8.52). Ken nonchalantly ignored the floor function.

Back to page 38 →

Additivity of entropy and information, the maximal contracting possible for a given bit chain, is based on the logarithm function and the factorization of probabilities for independent statistical events. it is easy to show that the entropy defined by Shannon is extensive, proportional to the code length.

additive information, from page 39

The average code length for doubles is

$$\langle \ell_2 \rangle \; = \; \sum_i p_i \ell_i \tag{8.54}$$

and by quadruples is

$$\langle \ell_4 \rangle \; = \; \sum_i p_i \sum_j p_j (\ell_i + \ell_j) \; = \; 2 \, \langle \ell_2 \rangle \, . \tag{8.55}$$

In general

$$\langle \ell_{2N} \rangle \; = \; \sum_{i_1} \cdots \sum_{i_N} p_{i_1} \cdots p_{i_N} \, (\ell_{i_1} + \ldots \ell_{i_N}) \; = \; N \, \langle \ell_2 \rangle \, . \tag{8.56}$$

Back to page 39 →

Natural languages are redundant, they are not comprised to their minimal length. It is necessary, for most communication happens via noisy channels. The frequency table of English letters in English texts is tabulated below, taken from the internet (link in the footnote). This will be used in some of the coming scenes.

English letter frequencies, from page 39

Occurrences of letters in 40.000 English words.[1]

Letter	Count	Frequency in %	Cumulative %
E	21912	12.02	12.02
T	16587	9.10	21.12
A	14810	8.12	29.24
O	14003	7.68	36.92
I	13318	7.31	44.23
N	12666	6.95	51.18
S	11450	6.28	57.46
R	10977	6.02	63.48
H	10795	5.92	69.40
D	7874	4.32	73.72
L	7253	3.98	77.70
U	5246	2.88	80.58
C	4943	2.71	83.29
M	4761	2.61	85.90
F	4200	2.30	88.20
Y	3853	2.11	90.31
W	3819	2.09	92.40
G	3693	2.03	94.43
P	3316	1.82	96.25
B	2715	1.49	97.74
V	2019	1.11	98.85
K	1257	0.69	99.54
X	315	0.17	99.71
Q	205	0.11	99.82
J	188	0.10	99.92
Z	128	0.07	99.99

Back to page 39 →

Extensivity of the Shannon entropy, based on its additivity for factorizing probabilities of coincident occurrences of independent events, is demonstrated below.

Shannon extensive, from page 40

For a joint couple probability of the form $p_{ij} = p_i q_j$, the Shannon entropy is additive, even if $p_i \neq q_i$. Here the factors are the marginal probabilities, the

[1]The data in the table is based on: Math Explorers' Club of the Department of Mathematics of the Cornell University (2003-2004). *English Letter Frequency (based on a sample of 40,000 words).* Department of Mathematics, Cornell University, Ithaca, NY. https://pi.math.cornell.edu/m̃ec/2003-2004/cryptography/subs/frequencies.html

probabilities irrespective of the state of the other

$$p_i \equiv \sum_j p_{ij} = p_i \sum_j q_j, \qquad q_j \equiv \sum_i p_{ij} = q_j \sum_i p_i, \tag{8.57}$$

provided that they are normalized. The entropy in this case is extensive

$$S_2 = -\sum_{ij} p_{ij} \log_2 p_{ij} = -\sum_i \sum_j p_i q_j \left(\log_2 p_i + \log_2 q_j\right) = 2S_1. \tag{8.58}$$

Back to page 40 →

Reality is not extensive. Additivity often can be established in systems consisting of many degrees of freedom, but not always. Santino demonstrates that the assumption of additivity and hence extensivity in natural languages leads to contradiction with facts.

contradicting to extensivity, from page 40

In this scenario $p_{EX} \approx p_{XE}$ and $p_{ET} \approx p_{TE}$, while definitely $p_{XT} \gg p_{TX}$. The product hypothesis contradicts to this.

From the equalities,

$$p_E q_X \approx p_X q_E, \qquad p_E q_T \approx p_T q_E \tag{8.59}$$

we derive the ratio

$$q_X / q_T \approx p_X / p_T. \tag{8.60}$$

On the other hand from the strong inequality we obtain

$$p_X q_T \gg p_T q_X \quad \to \quad q_X / q_T \ll p_X / p_T. \tag{8.61}$$

Back to page 40 →

Non-extensive entropies, as long as they are group entropies, use another function in place of the logarithm. This Csiszár form, used in the note below, also receives restrictions. Most prominently in the case of such PDF-s when only one state is occurring with probability one and all the others with probability zero must lead to zero entropy.

zero entropy PDF, from page 48

The distribution is like $p_i = \{1, 0, 0, \ldots\}$, being one for one state and zero for all others. The generalized trace entropy equals

$$S_f = \sum_{i=1}^{N} p_i f(1/p_i) = f(1). \tag{8.62}$$

Since this is a frozen state with zero degeneracy, it usually occurs at zero absolute temperature. Therefore according to the third law of thermodynamics it has to be zero: $f(1) = 0$.

This extra requirement is satisfied by the classical, Boltzmann-Gibbs entropy, and by all Rényi and Tsallis entropies.

A caveat: this works only if the asymptotics $\lim_{x\to 0} x f(1/x) = 0$ is also satisfied. It is true for $q > 0$, the same condition as the one for $f'' < 0$.

Back to page 48 →

An unexpected use of the convexity inequality is a possible "ranking" between entropies. It is particularly interesting to check any given entropy formula on the uniform distribution, where it reaches its maximum. The short derivation below reveals that the entanglement entropy is a special case of the Tsallis entropy.

ranking of entropies, from page 48

The convexity inequality can be used also in a way to get ranking between the different entropy formulas. Taking $f(x) = \ln x$, (with $f'' < 0$) and $x_i = p_i$ we obtain

$$\sum_i p_i \ln p_i \;\leq\; \ln\left(\sum_i p_i^2\right). \tag{8.63}$$

Its negative provides an information on the Boltzmann-Gibbs entropy as follows:

$$S_{\mathrm{BG}} \;=\; -\sum_i p_i \ln(p_i) \;\geq\; -\ln\left(\sum_i p_i^2\right) \;=\; S_{\mathrm{R,q=2}}. \tag{8.64}$$

The corresponding Tsallis entropy has a simple form, too:

$$S_{\mathrm{T,q=2}} \;=\; \frac{\mathrm{e}^{(1-2)S_{\mathrm{R,q=2}}} - 1}{1-2} \;=\; \sum_i p_i(1-p_i). \tag{8.65}$$

The latter is obviously zero at the singular PDF, and so are the Rényi and BG versions. In this case the equality holds.

Back to page 48 →

The following derivation investigates the Gini index, characterizing wealth and income inequality. It will be related to entropy formulas, discussed so far, later. Certainly, it is presented by Vilmano.

Gini index definition, from page 48

Let x and y be two random values taken from the *same* PDF, $P(x)$. The Gini index measures the expected difference, compared to the expected sum:

$$G = \frac{\langle |x-y| \rangle}{\langle |x+y| \rangle}. \tag{8.66}$$

As it is, we have a ratio of two expectation values. Gini wanted to ensure that this measure is always between zero and one. While it is true that $|x-y| \leq |x+y|$ for any $xy \geq 0$, it is not automatic for the ratio of the expectation values. Only if both values, x and y, are positive.
This definition is commonly used in another form, applied to nonnegative values of x and y (taken from the same set). In this case trivially $|x+y| = x+y$ and $\langle x+y \rangle = 2\langle x \rangle$. For the difference one distinguishes between $y \leq x$ and $y \geq x$ cases. But then, since the values are taken from the same distribution, the role of x and y can be interchanged. It suffices to consider, say, the $y \geq x$ case, and to write

$$G = \frac{1}{\langle x \rangle} \int_0^\infty dx \int_x^\infty dy \, (y-x) P(x) P(y). \tag{8.67}$$

Obviously for $P(x) = \delta(x-a)$, a singularly narrow distribution, we have $G = 0$.

Back to page 48 →

Non-Boltzmannian entropy formulas contain parameters describing the class of functions applied in the Csiszár form. In physics the question of their meaning in terms of real world parameters frequently asked. In particular the Tsallis and Rényi entropy formulas, for short sometimes called "q-entropies", look mysterious. The following derivation relates the parameter q to thermodynamical concepts, in particular to phase space volume ratios. It turns out that while the absolute temperature, T, measures average of the energy per degree of freedom (i.e. phase space dimension), the q is related to the variance.

q related to variation and finite size, from page 60

If we pick up a subsystem of a bigger one, even a single atom or particle, we look for the probability that it has a certain energy, ε, given the total energy, E. The non-trivial factor in the probability of ε is the ratio of phase space volumen parts corresponding to the rest[2], $V(E-\varepsilon)/V(E)$.

Now we follow Boltzmann and take $V(E) \propto e^{S(E)}$. In addition we average over parallel ensembles, realizations of the same physical body. Then our definition becomes

$$\rho(\varepsilon) \equiv \left\langle e^{S(E-\varepsilon)-S(E)} \right\rangle. \tag{8.68}$$

[2]Santino assumes for short that the energy is the lone quantity determining these ratios. In fact the dimensionality of phase space is influenced by the number of degrees of freedom, too. For large numbers we may neglect these differences.

Usually one expands the $S(E)$ function describing the equation of state until the linear terms. Here we expand until the quadratic term, and also expand the exponential,

$$\rho(\varepsilon) = \left\langle 1 - \varepsilon S'(E) + \frac{\varepsilon^2}{2}\left(S'(E)^2 + S''(E)\right) + \ldots \right\rangle . \tag{8.69}$$

We interpret the expectation values as $\beta = \langle S'(E)\rangle$, and then $\left\langle S'(E)^2 \right\rangle = \beta^2 + \Delta\beta^2$, together with $\langle S''(E)\rangle = d\beta/dE = -\beta^2/C$, containing the total heat capacity, C, usually linear in the system size. We compare this result with the expansion of the Tsallis–Pareto PDF up to second order,

$$\left(1 + (q-1)\frac{\varepsilon}{T}\right)^{-\frac{1}{q-1}} = 1 - \frac{1}{T}\varepsilon + \frac{q}{2T^2}\varepsilon^2 + \ldots \tag{8.70}$$

Comparing term by term we conclude at the following meaning of the Tsallis parameters T and q:

$$T = \frac{1}{\beta}, \qquad q = 1 + \frac{\Delta\beta^2}{\beta^2} - \frac{1}{C}. \tag{8.71}$$

Back to page 60 →

Following the above interpretation of T and q as thermodynamical parameters, another problem arises: is it possible and how to reach $q = 1$? That will be done by replacing the original entropy by another quantity, its additive formal logarithm. The derivation below demonstrates that for any group entropy, by definition associative, a common function, $K(S)$, does this job.

Associativity and formal logarithm, from page 61

Let us denote the entropy of a composed object from two subsystems with S_1 and S_2 by $S_1 \oplus S_2$. Associativity, a basic property of all group entropies, prescribes

$$(S_1 \oplus S_2) \oplus S_3 = S_1 \oplus (S_2 \oplus S_3). \tag{8.72}$$

Imagine that the operation $\oplus$ is quite general. Then three functions are involved in its definition:

$$P(S_1 \oplus S_2) = L(S_1) + R(S_2), \tag{8.73}$$

the pair function, P (must be invertible), the left and right element functions, L and R, respectively.

Now, the associativity expressed in eq. (8.72) requires

$$\begin{aligned} &P^{-1}\left(L[P^{-1}(L(S_1) + R(S_2))] + R(S_3)\right) \\ &\qquad = P^{-1}\left(L(S_1) + L[P^{-1}(L(S_2) + R(S_3))]\right). \end{aligned} \tag{8.74}$$

This can be if $L \circ P^{-1} = \mathrm{id}$, i.e. the functions L and P are identical. But then also follows from the above that

$$L(S_1) + R(S_2) + R(S_3) \; = \; L(S_1) + L(S_2) + R(S_3), \tag{8.75}$$

so also L and R are identical functions. This solely function in the background is the formal logarithm, so

$$K(S_1 \oplus S_2) \; = \; K(S_1) \; + \; K(S_2). \tag{8.76}$$

Back to page 61 →

Based on the previous, $K(S)$ can be used instead of S, the former being by construction additive. This deforms the calculation of phase space volumes and the sought environmental-statistical factor will contain a q_K parameter. At the end the problem can be inverted: what $K(S)$ function and by that what group entropy, will lead to $q_K = 1$, to the most Boltzmannian description?

$K(S)$ based non-additivity parameter, from page 61

We follow the phase space volume model with the new assumption, $V(E) \sim \mathrm{e}^{K(S(E))}$. Then our statistical factor from the phase space ratio can be expanded for small subsystem energy as follows

$$\left\langle \mathrm{e}^{K(S(E-\varepsilon))-K(S(E))} \right\rangle \; = \; \left\langle \mathrm{e}^{K(S-\varepsilon S' + \frac{\varepsilon^2}{2} S'')-K(S)} \right\rangle \tag{8.77}$$

Expanding also the $K(S)$ function and the exponential up to second order in ε, we arrive at the following comparison with the Tsallis–Pareto distribution:

$$\left\langle 1 - \varepsilon K'S' + \frac{\varepsilon^2}{2}\left(S''K' + (S')^2(K'' + (K')^2)\right) \right\rangle \; = \; 1 - \frac{\varepsilon}{T_K} + \frac{\varepsilon^2}{2T_K^2} q_K \tag{8.78}$$

with altered parameters T_K and q_K. We extract q_K from the term by term comparison and obtain

$$q_K \; = \; \frac{1}{K'} \frac{\langle S'' \rangle}{\langle S' \rangle^2} + \left(1 + \frac{K''}{(K')^2}\right) \frac{\langle (S')^2 \rangle}{\langle S' \rangle^2}. \tag{8.79}$$

Finally using the expectation values, $\langle S' \rangle = \beta$, $\langle (S')^2 \rangle = \beta^2 + \Delta\beta^2$ and $\langle S'' \rangle =$

$d\beta/dE = -\beta^2/C$, together with the former definition of the paremeter q, we arrive at

$$q_K = \left(q + \frac{1}{C}\right)\left(1 + \frac{K''}{(K')^2}\right) - \frac{1}{C}\frac{1}{K'}. \tag{8.80}$$

Back to page 61 →

Before discussing the general result, for the sake of deeper understanding, some extreme limiting cases will be discussed in what follows. Fluctuation dominance means, that the size of the environment is taken in the so called thermodynamical limit, assuming infinite heat capacity, but the variation between instances of the ensemble are kept finite. This case resembles a Tsallis–Abe type of non-additivity.

Fluctuation dominance, from page 62

We assume that the variance in the circumstances, $\Delta\beta^2/\beta^2$, is finite while the heat capacity is infinite. Then my formula above reduces to

$$q_K = q\left(1 + \frac{K''}{(K')^2}\right). \tag{8.81}$$

Re-arranged we obtain

$$\left(\frac{1}{K'}\right)' = -\frac{K''}{(K')^2} = 1 - \frac{q_K}{q} = \alpha. \tag{8.82}$$

With the S-independent parameter α it is easy to solve this differential equation. We fix $K'(0) = 1$ and $K(0) = 0$ and arrive at

$$K(S) = \frac{1}{\alpha}\ln(1 + \alpha S). \tag{8.83}$$

Here $\alpha = \frac{\Delta\beta^2}{\beta^2 + \Delta\beta^2}$ takes values between zero and one. The resolution of the composition law in eq.(8.76) becomes

$$S_1 \oplus S_2 = S_1 + S_2 + \alpha S_1 S_2. \tag{8.84}$$

Back to page 62 →

Once having a general formal logarithm entropy, $K(S)$, the entropy – probability formula is naturally based on it. No other choice remains. In the above discussed fluctuation dominated case a new formula arises, with logarithm of the logarithm.

deformed entropy – probability formula, from page 62

In the original formula the logarithm was chosen to ensure additivity for factorizing probabilities,

$$-\ln p_{ij} \;=\; -\ln p_i \,-\ln p_j, \tag{8.85}$$

its generalization uses the same property of the formal logarithm belonging to an arbitrary (but associative) composition rule:

$$K(-\ln p_{ij}) \;=\; K(-\ln p_i) \,+\, K(-\ln p_j). \tag{8.86}$$

By doing so, the generalized entropy of a large composed system is defined as the expectation value of such terms:

$$K(S) \;\equiv\; \sum_i p_i K(-\ln p_i). \tag{8.87}$$

In fact this is additive for the factorizing joint probability:

$$\begin{aligned} K(S_{12}) \;&=\; \sum_i \sum_j p_{ij} K(-\ln p_{ij}) \\ &=\; \sum_i p_i K(-\ln p_i) \sum_j p_j \,+\, \sum_i p_i \sum_j p_j K(-\ln p_j) \\ &=\; K(S_1) + K(S_2). \end{aligned} \tag{8.88}$$

In the previous special case of an infinitely large heat container with still appreciable finite temperature fluctuations we have

$$K(S) \;=\; \frac{1}{\alpha} \ln(1+\alpha S) \;=\; \sum_i \frac{p_i}{\alpha} \ln(1-\alpha \ln p_i). \tag{8.89}$$

Note that for the dominant states p_i is near to one, so $-\ln p_i$ is a small number and the contributing term is near to the classical one, $-p_i \ln p_i$.

Back to page 62 →

And now the general case follows. General, however, assuming that ensemble fluctuations and the total heat capacity are independent of the entropy. Given the expectation value and variance in the inverse thermodynamical temperature, $S'(E)$, together with the total heat capacity a two-parameter entropy formula can be derived. This includes the Tsallis entropy as a special case, whose particular case in turn is the Boltzmann entropy.

General deformed entropy with constant parameters, from page 63

We start with eq.(8.80) and use the notation $F(S) = 1/K'(S)$. Then we have to solve

$$q_K \;=\; \left(1+\frac{\Delta\beta^2}{\beta^2}\right)\left(1-F'\right) \;-\; \frac{1}{C}\,F, \tag{8.90}$$

with $F(0) = 1$. For entropy-independent fluctuation measure and heat capacity this can be done in analytic form. With known S-depedence it also can be solved by quadrature, i.e. by integrating and exponentiating known functions. Let us take the first case, when $\mu = \Delta\beta^2/\beta^2$ and $\lambda = 1/C$ are parameters, not functions of the entropy, S. Also we aim at having $q_K = 1$ for retaining additivity for $K(S)$. Then we solve

$$(1+\mu)F' + \lambda F = \mu. \tag{8.91}$$

The solution is standard,

$$F(S) = \frac{\mu}{\lambda} + \left(1 - \frac{\mu}{\lambda}\right) \mathrm{e}^{-\frac{\lambda S}{1+\mu}}. \tag{8.92}$$

From here using $K'(S) = 1/F(S)$ and $K(0) = 0$ one obtains

$$K(S) = \frac{1+\mu}{\mu} \ln\left(1 - \frac{\mu}{\lambda} + \frac{\mu}{\lambda}\mathrm{e}^{\frac{\lambda S}{1+\mu}}\right). \tag{8.93}$$

Finally it produces a generalized entropy – probability relation as follows

$$K(S) = \frac{1+\mu}{\mu} \sum_i p_i \ln\left(1 - \frac{\mu}{\lambda} + \frac{\mu}{\lambda} p_i^{-\frac{\lambda}{1+\mu}}\right). \tag{8.94}$$

Back to page 63 →

A missing piece in the general result is its relation to the q parameter of the Tsallis entropy. This connection is revealed in the next paragraph.

the missing piece, from page 63

Well, we use the fact that $\ln(1+x) \approx x$ for $x \ll 1$. Then

$$K(S) = \frac{1+\mu}{\lambda} \sum_i \left(p_i^{1-\frac{\lambda}{1+\mu}} - p_i\right), \tag{8.95}$$

which *is* the Tsallis' q-entropy formula with

$$1 - q_T = \frac{\lambda}{1+\mu} = \frac{1}{C} \frac{\beta^2}{\beta^2 + \Delta\beta^2} \tag{8.96}$$

But $\mu \ll \lambda$ also means $\Delta\beta^2 \ll \beta^2$, so in the same approximation

$$q_T = 1 - \frac{1}{C} = q. \tag{8.97}$$

Back to page 63 →

In Ludvick's dream a world appears where scientists believe that continuous models would describe facts. The formulas below demonstrate that one cannot uniquely relate the continuous to the discrete.

Discrete vs continuous PDF, from page 70

A normal probability density function, a pdf, is a continuous nowhere negative function, $P(x) \geq 0$, normalized by its integral over all possible values of x:

$$\int P(x)\, dx \;=\; 1. \tag{8.98}$$

We obtain a discrete version of it by dividing the integration range into finite bins between the points x_i and x_{i+1}. The determinate integrals

$$p_i \;\equiv\; \int\limits_{x_i}^{x_{i+1}} P(x)\, dx, \tag{8.99}$$

are then countably many values, those sum is normalized

$$\sum_i p_i \;=\; 1. \tag{8.100}$$

Any expectation value, and the entropy is also such a construction, is defined as an integral with the original, the "real" PDF:

$$\langle f \rangle \;\equiv\; \int f(x) P(x)\, dx. \tag{8.101}$$

This can never be equal to

$$\overline{f} \;\equiv\; \sum_i p_i f_i, \tag{8.102}$$

there will always be corrections in the order of some power of the bin length, whatever way the f_i values are constructed. Certainly $f(x_i)$ or $f(x_{i+1})$ won't do, and their linear combinations also leave some deviance, although possibly at higher order.

Back to page 70 →

Master equations describe the time evolution of probability distributions. Stationary solutions to stochastic processes may lead to different results in the discrete than in the continuous model. Below we compare these two types of master equations in general.

Discrete vs continuous master equation, from page 73

The n-th element in a discrete set of states has the probabilty P_n, i.e. the occurrence frequency in a huge sample fluctuates around this value. Imagine we measure various realizations of the same system, with PDF-s developing in time, according to the simple master equation

$$\dot{P}_n = \sum_m (w_{nm} P_m - w_{mn} P_n). \qquad (8.103)$$

Here the overdot denotes the time derivative and w_{mn} is the transition rate from a state n to m.

For a stationary distribution, Q_n, it suffices if the total balance condition is fulfilled

$$\sum_m (w_{nm} Q_m - w_{mn} Q_n) = 0. \qquad (8.104)$$

These are as many equations as states, so the stationary Q_n-s are uniquely determined. Note that the *detailed balance* condition on the other hand requires vanishing of all terms in the above sum! That means restricting conditions on the transition matrix elements, w_{mn}. For the uniform distribution, usually assumed in lack of any specific information, this restriction is simply the *micro-reversibility*: $w_{mn} = w_{nm}$.

All these equations can also be written in a continuous model. It suffices to look at the dynamical master equation:

$$\dot{P}(x) = \int dy \, [w(x,y)P(y) - w(y,x)P(x)]. \qquad (8.105)$$

Back to page 73 →

Here a special model, the Local Growth Global Reset (LGGR) model is presented. Its discrete and continuous versions are displayed.

Local Growth Global Reset Model, from page 73

In this model only two types of transitions are allowed in a time step: i) from n to $n+1$, this is the local growth process, and ii) from any state n to 0, this is the global reset process. We have

$$w_{nm} = \mu_m \delta_{n,m+1} + \gamma_m \delta_{n,0}. \qquad (8.106)$$

It automatically follows that the rate of the reverse process is

$$w_{mn} = \mu_n \delta_{m,n+1} + \gamma_n \delta_{m,0}. \qquad (8.107)$$

The master equation describing the evolution of the PDF becomes

$$\dot{P}_n = \mu_{n-1} P_{n-1} - (\mu_n + \gamma_n) P_n + \langle \gamma \rangle \, \delta_{n,0}. \qquad (8.108)$$

Neglecting the equation for $\dot{P}_0$, since P_0 always can be reconstructed form the normalized sum, $\sum_n P_n = 1$, the continuous analog model is given by

$$\frac{\partial}{\partial t} P(x,t) \;=\; -\frac{\partial}{\partial x}\left(\mu(x)P(x,t)\right) - \gamma(x)P(x,t). \tag{8.109}$$

We compare the stationary solutions to these systems in two simple cases.

Back to page 73 →

Stationary distributions of the LGGR model with constant rates are exponential, but with different parameters in the discrete and continuous case.

Constant LGGR rates, from page 74

In such a case the LGGR transition rates are constants, $\mu_n = \sigma$, $\gamma_n = \gamma$. The stationary PDF satisfies

$$\sigma Q_{n-1} - (\sigma+\gamma)Q_n \;=\; 0. \tag{8.110}$$

The solution is obtained recursively

$$Q_n \;=\; \frac{\sigma}{\sigma+\gamma}\, Q_{n-1} \;=\; \left(\frac{\sigma}{\sigma+\gamma}\right)^n Q_0. \tag{8.111}$$

In essence this an exponential (or geometrical) distribution,

$$Q_n \;=\; Q_0 \mathrm{e}^{-\beta\, n} \qquad \text{with} \qquad \beta \;=\; \ln\left(1+\gamma/\sigma\right). \tag{8.112}$$

On the other hand the continuous model delivers the following constraint on the stationary distribution:

$$-\frac{\partial}{\partial x}(\sigma Q) \;-\; \gamma Q \;=\; 0, \tag{8.113}$$

leading to the solution

$$Q(x) \;=\; Q(0)\mathrm{e}^{-\beta\, x} \qquad \text{with} \qquad \beta \;=\; \gamma/\sigma. \tag{8.114}$$

Back to page 74 →

Stationary solutions in the LGGR model with constant reset rate but a preferential local growth rate are power-law tailed. Here the discrete and continuous versions produce different functional forms for the analytic solutions. The section below presents the discrete case, leading to the Waring distribution.

LGGR Waring, from page 75

The linear preference in the growth rate is described by $\mu_n = \sigma(n+b)$, usually with the $b=1$ choice. The reset rate is still constant, $\gamma_n = \gamma$. The recursion for the stationary PDF leads to the following result

$$Q_n \;=\; Q_0 \, \frac{b(b+1)\ldots(b+n-1)}{(b+1+\gamma/\sigma)\ldots(b+\gamma/\sigma+n)} \tag{8.115}$$

Actually the n-term product has a compact notation, $(b)_n = b(b+1)\ldots(b+n-1)$ is called a Pochhammer symbol. Using this and $c = b+1+\gamma/\sigma$ the simple answer is a Waring distribution:

$$Q_n \;=\; Q_0 \, \frac{(b)_n}{(c)_n}. \tag{8.116}$$

The high-n tail of this distribution is a power-law:

$$Q_n \;\to\; Q_0 \, n^{b-c} \;=\; \text{const.}\, n^{-1-\gamma/\sigma}. \tag{8.117}$$

Back to page 75 →

The preferential growth rate in the continuous model leads to the Pareto distribution – observed in real world income data, in popularity statistics and in transverse momentum spectra of hadrons produced in high-energy accelerator experiments.

LGGR Pareto, from page 75

The growth rate, $\mu(x)$ and the reset rate $\gamma(x)$ define a first order differential equation for the stationary PDF:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mu(x)Q(x)\right) \;=\; -\gamma(x)Q(x). \tag{8.118}$$

Such problems can always be solved by quadrature, by expressing the result via integrals. We have

$$Q(x) \;=\; \frac{\mu(0)}{\mu(x)}\, Q(0)\, \mathrm{e}^{-\int\limits_0^x \frac{\gamma(u)}{\mu(u)} du} \tag{8.119}$$

For the linear preference in the growth rate, $\mu(x) = \sigma(x+b)$, with state-independent (democratic) reset rate, $\gamma(x) = \gamma$, we obtain

$$Q(x) \;=\; Q(0)\left(1+\frac{x}{b}\right)^{-1-\gamma/\sigma}. \tag{8.120}$$

Back to page 75 →

Income distribution and popularity data have been fitted also by log-normal distributions. The derivation below reconstructs the reset rate necessary for this result in the LGGR model, given a linear preferential growth rate.

log-normal, from page 75

We start with the log-normal distribution as stationary PDF:

$$Q(x) \;=\; \frac{K}{x+b}\,\mathrm{e}^{-[\ln(x+b)-a]^2/2\alpha}. \tag{8.121}$$

We keep the growth rate in the linear preferential form: $\mu(x) = \sigma(x+b)$. It is necessary to choose the same offset b. Now we obtain the derivative of the product $\mu(x)Q(x)$:

$$-\frac{\mathrm{d}}{\mathrm{d}x}(\mu Q) \;=\; \sigma K \mathrm{e}^{-[\ln(x+b)-a]^2/2\alpha}\,\frac{\ln(x+b)-a}{\alpha}\,\frac{1}{x+b}. \tag{8.122}$$

From here we obtain the necessary reset rate as being

$$\gamma(x) \;=\; -\frac{1}{Q}\frac{\mathrm{d}}{\mathrm{d}x}(\mu Q) \;=\; \frac{\sigma}{\alpha}\left[\ln(x+b)-a\right]. \tag{8.123}$$

Back to page 75 →

The same stationary PDF can be obtained by assuming various growth and reset rates; only their combination is fixed by observing distribution data. The demonstration of various rates leading to the same distribution, as a stationary one, follows.

various rates with the same stationary PDF, from page 75

For obtaining the same $Q(x)$, one needs the same function

$$\varphi(x) \;=\; \frac{\gamma(x)+\mu'(x)}{\mu(x)} \;=\; -\frac{1}{Q}\frac{\mathrm{d}Q}{\mathrm{d}x}. \tag{8.124}$$

Any pair of income dependent rates for a little growth, $\mu(x)$ and a one-step ruin, $\gamma(x)$, related with the same $\varphi(x)$ function result in the same stationary PDF. Fixing one of the pairs, of course, determines the other – together with the known $Q(x)$.

In general we have a reset rate of

$$\gamma(x) \;=\; \varphi(x)\mu(x) \;-\; \mu'(x) \;=\; -\frac{1}{Q}\frac{\mathrm{d}}{\mathrm{d}x}(\mu Q). \tag{8.125}$$

In particular with constant growth rate, $\mu(x) = \mu$, the necessary reset rate is proportional to the negative slope, $\varphi(x)$, in eq.(8.124). With linear growth rate, $\mu(x) = \sigma(x+b)$, one gets $\gamma(x) = \sigma[(x+b)\varphi(x) - 1]$.

Back to page 75 →

In case of particle spectra most probably the fluctuating number of produced hadrons, causing a fluctuation in the dimensionality of the phase space is responsible for the occurrence of a statistical Pareto distribution. This possibility is shown in the next section.

Fluctuating Phase Space Dimensions, from page 83

Given this situation, we look for the statistical factor as the ratio of phase space volumes corresponding to the energy $E-\epsilon$ and E. For ideal gases this is a power of the energy ratio, the power being the dimension of the total phase space – following strongly the number of actors.

Now we consider that this very number, N, has a distribution, and write

$$\rho(\epsilon) = \sum_{N=0}^{\infty} P_N \left(1 - \frac{\epsilon}{E}\right)^N . \tag{8.126}$$

For a Poisson distribution of N, this is simply the Boltzmann exponential with the temperature parameter $T = E/\langle N \rangle$:

$$\rho(\epsilon) = \sum_{N=0}^{\infty} \frac{\langle N \rangle^N}{N!} \mathrm{e}^{-\langle N \rangle} \left(1 - \frac{\epsilon}{E}\right)^N = \mathrm{e}^{-\langle N \rangle \epsilon / E}. \tag{8.127}$$

On the other hand with the negative binomial distribution, NBD, the one-particle energy is distributed as the Pareto-law:

$$\begin{aligned} \rho(\epsilon) &= \sum_{N=0}^{\infty} \binom{N+K}{N} f^N (1+f)^{-N-K-1} \left(1 - \frac{\epsilon}{E}\right)^N \\ &= \left(1 + \frac{\langle N \rangle}{E} \frac{\epsilon}{K+1}\right)^{-K-1} . \end{aligned} \tag{8.128}$$

For the NBD we had $\langle N \rangle = f(K+1)$. This is also a Tsallis distribution with parameters $T = E/\langle N \rangle$ and $q = 1 + 1/(K+1)$.

Back to page 83 →

In the next part the Bernoulli distribution is presented. That distribution emerges from "either-or" processes with an average choice probability of f being between zero and one. It is alike the Heads or Tails game. The name of "binomial" is also explained by expanding a "binom", a sum of two parts, 1 and x, on an arbitrary power.

Bernoulli and Pascal, from page 84

The Bernoulli distribution is based on the expansion of the power of a two-part expression, the binom. In its simplest form it describes the K-th power of 1 and x:

$$(1+x)^K \;=\; \sum_{N=0}^{\infty} \binom{K}{N} \, x^N. \tag{8.129}$$

Here the summation may run to infinity, since for any $N > K$ it is true that

$$\binom{K}{N} = \frac{K!}{N!(K-N)!} = 0,$$

due to the factorial of a negative number in the denominator.

These binomial coefficients constitute the Pascal triangle, according to the factors in the terms for $K = 0, 1, 2, 3 \ldots$ They are: (1), $(1,1)$, $(1,2,1)$, $(1,3,3,1)$ and so on. Written line by line each element is the sum of the two neighbours in the line one above.

Applying this formula to $x = f/(1-f)$, a ratio of particles to holes, or more generally a ratio of hits to misses, when the probability for a single hit is f, we obtain the normalized sum

$$(f+1-f)^K \;=\; 1 \;=\; \sum_{N=0}^{\infty} \binom{K}{N} \, f^N \, (1-f)^{K-N}. \tag{8.130}$$

Here the summands deliver the probability of N hits from K probes at the individual probabilities being f. This is the Bernoulli distribution.

Back to page 84 →

The negative binomial expansion and distribution (NBD) are presented here. This roots in the generalization of the factorial product for negative numbers – whence is the name.

NBD version, from page 84

We apply formally the above discussed traditional binomial expansion instead of K to $-(K+1)$ and instead of x to $-x$:

$$(1-x)^{-K-1} \;=\; \sum_{N=0}^{\infty} \binom{-K-1}{N} \, (-x)^N. \tag{8.131}$$

Now, please observe with me that

$$\binom{-K-1}{N} = \frac{(-K-1)(-K-2)\dots(-K-N)(-K-N-1)\dots}{N!\,(-K-N-1)(-K-N-2)\dots} \tag{8.132}$$

Here infinitely many negative factors cancel and in the numerator exactly N factors remain. Therefore we write

$$\begin{aligned}\binom{-K-1}{N} &= (-1)^N \frac{(K+1)(K+2)\dots(K+N)}{N!} \\ &= (-1)^N \frac{(K+N)!}{K!\,N!} = \binom{N+K}{N}(-1)^N. \end{aligned} \tag{8.133}$$

Finally we have the expansion of the negative powers as follows:

$$(1-x)^{-K-1} = \sum_{N=0}^{\infty} \binom{K+N}{N} x^N. \tag{8.134}$$

The negative binomial distribution (NBD) is obtained by using $x = f/(1+f)$ and multiplying the above expression with $(1+f)^{-K-1}$. The NBD probability is given by the terms in the sum:

$$P_N = \binom{K+N}{N} f^N (1+f)^{-K-1-N}. \tag{8.135}$$

Back to page 84 →

The Pascal triangle is certainly well known for the binomial expansion. Here its pendant for the negative binonmial expansion is displayed.

NBD Pascal triangle, from page 85

We write the negative powers each under the previous one and expand:

$$\begin{aligned}
K=0: &\quad (1-x)^{-1} &= 1+1x+1x^2+1x^3+1x^4+1x^5+\dots \\
K=1: &\quad (1-x)^{-2} &= 1+2x+3x^2+4x^3+5x^4+\dots \\
K=2: &\quad (1-x)^{-3} &= 1+3x+6x^2+10x^3+\dots \\
K=3: &\quad (1-x)^{-4} &= 1+4x+10x^2+\dots \\
K=4: &\quad (1-x)^{-5} &= 1+5x+\dots \\
K=5: &\quad (1-x)^{-6} &= 1+\dots
\end{aligned} \tag{8.136}$$

Back to page 85 →

The exponential dilution of the sample space acts as if a reset rate would be in effect for the dynamics of the probability distribution. We exemplify this correspondence for the scientific citation statistics, obtaining again a version of the LGGR model.

Exponential dilution, from page 88

This could be simpler to understand in the discrete model, resulting in a Waring distribution. But then the continuous version works by the same mechanism.

We have N_i citations for the i-th data set (individual or institution or a group). Having the μ_i local growth rate for increasing the number by one, once having already N_i, we model the dynamics as

$$\frac{dN_i}{dt} = \mu_{i-1}N_{i-1} - \mu_i N_i. \tag{8.137}$$

Furthermore, let the sum of all citations, as well as the sum of all publications, Facebook placements, etc, grow exponentially:

$$N(t) \equiv \sum_i N_i = N_0 e^{\gamma t}. \tag{8.138}$$

Then our model for the relative share, $P_i = N_i/N$, becomes

$$\frac{dP_i}{dt} = \frac{1}{N}\frac{dN_i}{dt} - \frac{N_i}{N^2}\frac{dN}{dt} = \mu_{i-1}P_{i-1} - (\mu_i + \gamma)P_i. \tag{8.139}$$

This effect, caused by the γ rate, we call *exponential dilution* of the data space.

Back to page 88 →

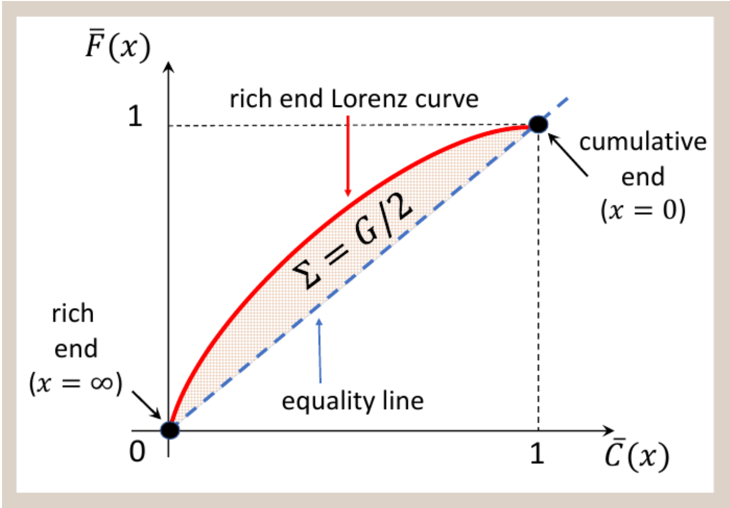
The mathematical presentation of the Lorenz curve, comparing two cumulative integrals of the wealth distribution, reveals its connection to the Gini index. The geometry is demonstrated in a figure drawing a Lorenz curve in the general case.

Lorenz curve, from page 96

The Lorenz curve relates the cumulative wealth, normalized by the average, and the cumulative population possessing it. I remind to the definitions:

$$\overline{C}(x) \equiv \int_x^\infty P(y)dy, \qquad \overline{F}(x) \equiv \frac{1}{\langle x \rangle}\int_x^\infty y\, P(y)dy. \tag{8.140}$$

The Lorenz curve is simply plotting $\overline{F}$ against $\overline{C}$. Since both cumulative quantities are normalized, the whole curve runs inside the unit square, comprising the whole distribution information in a finite cut. A typical curve is shown below.



Since both cumulative integrals are normalized, we have $\overline{C}(0) = 1$ and $\overline{F}(0) = 1$. Therefore at $x = 0$ we are at the right upper corner. Obviously, for $x \to \infty$, the Lorenz curve approaches the origo at the left lower corner.

Back to page 96 →

The area between the Lorenz curve and the diagonal is the half of the Gini index. Its proof is prepared by calculating areas as integrals.

Lorenz curve area, from page 96

Areas under and between curves are expressed by integrals. Denoting the half crescent shaped part by Σ it is obvious that

$$\int_0^1 \overline{F}\, d\overline{C} \;=\; \frac{1}{2} + \Sigma. \tag{8.141}$$

Looking from the left side you also recognize the inverse interpretation of the same curve and area as

$$\int_0^1 \overline{C}\, d\overline{F} \;=\; \frac{1}{2} - \Sigma. \tag{8.142}$$

Not only the trivial result, that their sum is unity, emerges, but also a remarkable expression for the sought area

$$2\Sigma \;=\; \int_0^1 \overline{F}\, d\overline{C} \;-\; \int_0^1 \overline{C}\, d\overline{F}. \tag{8.143}$$

Back to page 96 →

What has been shown about the Gini index earlier, here is used for the proof of the geometrical properties of the Lorenz curve here.

Gini index as area, from page 96

The integrals in terms of $\overline{F}$ and $\overline{C}$ are also expectation values, as follows:

$$\begin{aligned} \int_0^1 \overline{F} d\overline{C} &= \int_0^\infty \overline{F}(x) P(x) dx = \left\langle \overline{F} \right\rangle, \\ \int_0^1 \overline{C} d\overline{F} &= \int_0^\infty \overline{C}(x) \frac{x}{\langle x \rangle} P(x) dx = \frac{\left\langle x\overline{C} \right\rangle}{\langle x \rangle}. \end{aligned} \tag{8.144}$$

Their difference, as above, can be also written as double integrals

$$2\Sigma = \left\langle \overline{F} \right\rangle - \frac{\left\langle x\overline{C} \right\rangle}{\langle x \rangle} = \frac{1}{\langle x \rangle} \int_0^\infty dx \int_x^\infty dy\, (y - x)\, P(x) P(y) = G. \tag{8.145}$$

Back to page 96 →

Based on the Lorenz curve geometry we finally arrive at the definition of gintropy, representing the density of the half Gini index. The most relevant properties of this quantity are revealed, related to its interpretation via expectation values of cumulants.

Gintropy properties, from page 97

The definition is $\sigma = \overline{F} - \overline{C}$. But since trivially

$$\int_0^1 \overline{C}\, d\overline{C} = 1/2, \qquad \text{and} \qquad \int_0^1 \overline{F}\, d\overline{F} = 1/2, \tag{8.146}$$

do describe the area of triangles, the remaining integrals again lead to the Gini index:

$$\begin{aligned} \int_0^1 \sigma d\overline{C} &= \int_0^1 (\overline{F} - \overline{C}) d\overline{C} = (1/2 + \Sigma) - 1/2 = G/2, \\ \int_0^1 \sigma d\overline{F} &= \int_0^1 (\overline{F} - \overline{C}) d\overline{F} = 1/2 - (1/2 - \Sigma) = G/2. \end{aligned} \tag{8.147}$$

The integral of the gintropy either with $\overline{C}$, the cumulative population or with $\overline{F}$, the cumulative wealth, is equal to the half of the Gini index, G.

That the same quantity can be expressed with two different ways offers us a nice formula. We observe what the integrals over cumulatives mean in terms of the PDF, $P(x)$:

$$G/2 = \int_0^1 \sigma d\overline{C} = \int_0^\infty \sigma(x) P(x) dx = \langle \sigma \rangle ,$$

$$G/2 = \int_0^1 \sigma d\overline{F} = \int_0^\infty \sigma(x) \frac{x}{\langle x \rangle} P(x) dx = \frac{\langle x\sigma \rangle}{\langle x \rangle}. \tag{8.148}$$

The essence of this looks like a non-correlation formula

$$\langle x \cdot \sigma \rangle = \langle x \rangle \cdot \langle \sigma \rangle . \tag{8.149}$$

Back to page 97 →

Entropic style properties of gintropy, like non-negativity, convexity, location of its maximum are calculated.

Further gintropy properties, from page 97

First the gintropy is never negative. This can be seen by using the poor end cumulatives, $C = 1 - \overline{C}$ and $F = 1 - \overline{F}$, where it becomes necessary.

$$\sigma = \int_x^\infty \left(\frac{y}{\langle x \rangle} - 1 \right) P(y) dy = \int_0^x \left(1 - \frac{y}{\langle x \rangle} \right) P(y) dy. \tag{8.150}$$

For $x > \langle x \rangle$ the first, for $x < \langle x \rangle$ the second form is trivially positive, so we conclude that $\sigma(x) \geq 0$.

Second, $\sigma(x)$ has its maximum exactly at $x = \langle x \rangle$:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}x} = -\frac{x}{\langle x \rangle} P(x) + P(x), \tag{8.151}$$

so we have $d\sigma/dx = 0$ exactly at $x = \langle x \rangle$. It is a maximum since

$$\left. \frac{\mathrm{d}^2\sigma}{\mathrm{d}x^2} \right|_{d\sigma/dx=0} = -\frac{1}{\langle x \rangle} P\big(\langle x \rangle\big) < 0. \tag{8.152}$$

Finally, the gintropy as a function of the cumulative rich population is even more interesting. *It has a definite convexity!*

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\overline{C}} = \frac{x}{\langle x \rangle} - 1, \qquad \text{and} \qquad \frac{\mathrm{d}^2\sigma}{\mathrm{d}\overline{C}^2} = \frac{1}{\langle x \rangle} \frac{\mathrm{d}x}{\mathrm{d}\overline{C}} = -\frac{1}{\langle x \rangle \, P(x)} < 0. \tag{8.153}$$

Moreover, the same is true for it as a function of the cumulative wealth:

$$\frac{d\sigma}{d\overline{F}} = 1 - \frac{\langle x \rangle}{x}, \qquad \text{and} \qquad \frac{d^2\sigma}{d\overline{F}^2} = \frac{\langle x \rangle}{x^2}\frac{dx}{d\overline{F}} = -\frac{\langle x \rangle^2}{x^3 P(x)} < 0. \tag{8.154}$$

Back to page 97 →

The "natural" distribution is exponential, being inspired by the energy distribution of atoms in ideal gases. Such a distribution is taken to obtain the corresponding gintropy and Gini index.

Natural gintropy, from page 98

The natural distribution is exponential, like the Boltzmann-Gibbs one,

$$P(x) = \frac{1}{\langle x \rangle} e^{-x/\langle x \rangle}. \tag{8.155}$$

The cumulative population and wealth are the integrals

$$\overline{C} = \int_x^\infty P(y)dy = e^{-x/\langle x \rangle}, \quad \text{and} \quad \overline{F} = \int_x^\infty \frac{y}{\langle x \rangle} P(y)dy = \left(\frac{x}{\langle x \rangle} + 1\right) e^{-x/\langle x \rangle}. \tag{8.156}$$

The gintropy is given as a function of x and as a function of $\overline{C}$, as follows

$$\sigma = \frac{x}{\langle x \rangle} e^{-x/\langle x \rangle} = -\overline{C} \ln \overline{C}. \tag{8.157}$$

The Gini index becomes

$$G = \int_0^\infty \overline{C}(1-\overline{C}) \frac{dx}{\langle x \rangle} = \int_0^\infty \left(e^{-t} - e^{-2t}\right) dt = \frac{1}{2}. \tag{8.158}$$

Back to page 98 →

Income and wealth data in capitalistic (market) societies are best fitted by Tsallis–Pareto distributions. Below we analyze the corresponding gintropy and Gini index.

Capitalism gintropy, from page 98

We start with the cumulative rich,

$$\overline{C} = (1 + Ax)^{-B-1}. \tag{8.159}$$

The PDF is the negative derivative of $\overline{C}(x)$:

$$P(x) \;=\; -\frac{\mathrm{d}\overline{C}}{\mathrm{d}x} \;=\; A(B+1)\,(1+Ax)^{-B-2}. \tag{8.160}$$

We replace this result back into the definition of the cumulative wealth at the rich end and obtain

$$\overline{F} \;=\; \frac{A(B+1)}{\langle x\rangle} \int\limits_x^\infty (1+Ay)^{-B-2} y dy. \tag{8.161}$$

It is convenient to introduce the integration variable $t = (1 + Ay)$. Then $y = (t-1)/A$. After doing so we arrive at

$$\overline{F} \;=\; \frac{(B+1)}{A\,\langle x\rangle} \int\limits_{1+Ax}^\infty \left(t^{-B-1} - t^{-B-2}\right) dt. \tag{8.162}$$

The result of this integral is given by

$$\overline{F} \;=\; \frac{B+1}{A\,\langle x\rangle} \left[\frac{1}{B}(1+Ax)^{-B} - \frac{1}{B+1}(1+Ax)^{-B-1}\right] \tag{8.163}$$

Finally after some algebraic rearrangements of extracting common factors we arrive at

$$\overline{F} \;=\; \frac{1}{AB\,\langle x\rangle} \left[(B+1)(1+Ax) - B\right](1+Ax)^{-B-1}. \tag{8.164}$$

Knowing that $\overline{F}(0) = 1$ we have $\langle x\rangle = 1/(AB)$.

Here we recognize again $\overline{C}$ as a factor, and that leads us to yet another expression of the cumulative wealth:

$$\overline{F} \;=\; (B+1)\,\overline{C}^{\left(1-\frac{1}{B+1}\right)} \;-\; B\,\overline{C}. \tag{8.165}$$

Finally the gintropy, $\sigma = \overline{F} - \overline{C}$, as a function of $\overline{C}$ follows

$$\sigma \;=\; (B+1)\left[\overline{C}^{\frac{B}{B+1}} - \overline{C}\right]. \tag{8.166}$$

Using now the parameter $q = \frac{B}{B+1}$, we arrive at the Tsallis–form

$$\sigma \;=\; \frac{1}{1-q}\left[\overline{C}^{q} - \overline{C}\right]. \tag{8.167}$$

Back to page 98 →

The two class model sounds simple, but it already brings the most important parameters: the ratio of the income for the rich and poor, and the fraction of each, say the poor. All relevant quantities are calculated below.

Two class gintropy, from page 99

The PDF in a two class model is given as a mixture of two peaks,

$$P(x) \;=\; w\delta(x-a)+(1-w)\delta(x-b). \tag{8.168}$$

We assume that the lower income is a and the upper one is $b>a$. The relative fraction of the people in the lower class is w, in the upper class is $(1-w)$. This parameter is between zero and one.

The cumulative population is a two step staircase, being 100% for $x<a$, zero for $x>b$ and exactly $(1-w)$ in between:

$$\overline{C}(x) \;=\; w\Theta(a-x)+(1-w)\Theta(b-x). \tag{8.169}$$

It is also straightforward to obtain the cumulative income:

$$\overline{F}(x) \;=\; \frac{1}{\langle x\rangle}\left[w\int_x^{\infty} y\delta(y-a)dy \;+\; (1-w)\int_x^{\infty} y\delta(y-b)dy\right]. \tag{8.170}$$

The result of this integration is just a- and b-times the integrals by obtaining the cumulative population, respectively, and both divided by $\langle x\rangle$:

$$\overline{F}(x) \;=\; w\frac{a}{\langle x\rangle}\,\Theta(a-x) \;+\; (1-w)\frac{b}{\langle x\rangle}\,\Theta(b-x). \tag{8.171}$$

Since whenever $x<a$ it is also $x<b$, we can rewrite this result as

$$\overline{F} \;=\; \frac{wa+(1-w)b}{\langle x\rangle}\Theta(a-x) \;+\; \frac{(1-w)b}{\langle x\rangle}\,\Theta(x-a)\Theta(b-x). \tag{8.172}$$

From this form it is easy to see that due to $\overline{F}(0)=1$ we have

$$\langle x\rangle \;=\; wa+(1-w)b. \tag{8.173}$$

From these quantities the gintropy is constructed. It is nonzero only in the $a<x<b$ interval:

$$\sigma \;=\; \overline{F}-\overline{C} \;=\; w\Theta(a-x)\left[\frac{a}{\langle x\rangle}-1\right] \;+\; (1-w)\Theta(b-x)\left[\frac{b}{\langle x\rangle}-1\right]. \tag{8.174}$$

Whenever $x<a<b$, both conditions are fulfilled, and we arrive at

$$\sigma_{x<a} \;=\; \frac{wa+(1-w)b}{\langle x\rangle}-w-(1-w) \;=\; 0. \tag{8.175}$$

In the interval on the other hand only the second contribution survives:

$$\sigma_{x\in[a,b]} \;=\; (1-w)\left[\frac{b}{\langle x\rangle}-1\right]. \tag{8.176}$$

Trivially for $x > b$ none of the conditions, the arguments of the Theta-functions, are fulfilled therefore $\sigma_{x>b} = 0$.

A further statement is that in the interval $x \in [a,b]$, the constant value of gintropy is exactly the Gini index, G. For this purpose we use the half-moon area law relation and obtain

$$\frac{1}{2}G \;=\; \int_0^1 \sigma d\overline{C} \;=\; \sigma_{x\in[a,b]}\left[\overline{C}(a)-\overline{C}(b)\right]. \tag{8.177}$$

The values of $\overline{C}$ at the edge points $x = a$ and $x = b$ are best estimated by the mid value between those left and right from the points. Therefore

$$\overline{C}(b) \;=\; \frac{1}{2}\big(0+(1-w)\big) \;=\; \frac{1}{2}-\frac{w}{2}, \quad \text{and} \quad \overline{C}(a) \;=\; \frac{1}{2}\big(1+(1-w)\big) \;=\; 1-\frac{w}{2}. \tag{8.178}$$

From this $\overline{C}(a) - \overline{C}(b) = 1/2$ follows, so $\sigma_{x\in[a,b]} = G$.

Back to page 99 →

Beyond the gintropy formula reminding to the entanglement entropy, the two class income distribution is a theoretical tool to understand on what parameters the global inequality expressed in the Gini index depends on the strongest. Having enough difference between the income of the rich and the poor class, this index closes to the fraction of the poor.

Two-class Gini index, from page 99

Since the Gini index is the constant value of the gintropy function in the window $x \in [a,b]$, we have

$$G \;=\; (1-w)\left[\frac{b}{\langle x\rangle}-1\right]. \tag{8.179}$$

Utilizing that $\langle x\rangle = wa + (1-w)b$ we obtain

$$G \;=\; \frac{w(1-w)}{wa+(1-w)b}\,(b-a). \tag{8.180}$$

Back to page 99 →

Finally, for theoreticians in society construction, an eco-window case is studied. Here all incomes are equally probable or frequent between two fixed limits. This can also be achieved by strong commitment of the state into the economy: social support can cut from below, progressive taxation from above the income distribution.

Eco-window gintropy, from page 99

The PDF of this window-restricted income model is given as

$$P(x) \;=\; \frac{1}{b-a}\,\Theta(x-a)\,\Theta(b-x). \tag{8.181}$$

It is identically zero outside of the allowed interval. Obviously $\langle x \rangle = (a+b)/2$. The cumulative rich population function becomes

$$\overline{C}(x) \;=\; \begin{cases} 1, & x < a \\ \frac{b-x}{b-a}, & x \in (a,b) \\ 0, & b < x \end{cases} \tag{8.182}$$

The cumulative wealth, $\overline{F}(x)$ can be obtained in the nontrivial interval easily:

$$\overline{F}(x) \;=\; \frac{1}{(b-a)\,\langle x \rangle} \int\limits_x^b y dy \;=\; \frac{b^2-x^2}{b^2-a^2}. \tag{8.183}$$

In the $x < a$ region $\overline{F} = 1$, just continuing the value of the above from $x = a$. So finally the gintropy function is zero outside of the eco-window, and inside it becomes

$$\sigma(x) \;=\; \frac{b^2-x^2}{b^2-a^2} - \frac{b-x}{b-a} \;=\; \frac{(x-a)(b-x)}{b^2-a^2}. \tag{8.184}$$

Since inside the window $\overline{C}$ is a linear function of x, it is easy to express the above result in terms of this cumulative quantity

$$\sigma \;=\; \frac{b-a}{b+a}\,\overline{C}\,(1-\overline{C}). \tag{8.185}$$

Utilizing finally the half Gini index formula as the integral of $\sigma(\overline{C})$, we arrive at the alternative expression

$$\sigma \;=\; 3G\,\overline{C}\,(1-\overline{C}) \qquad \text{with} \qquad G \;=\; \frac{1}{3}\,\frac{b-a}{b+a}. \tag{8.186}$$

Back to page 99 →

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This book is an experiment in the genre of textbooks: mathematical formulas related to entropy, distributions, and inequality measures like the Gini index are embedded into a fictious story about seven scientists, meeting, and acting according to their respective characters in an imaginary project. The embedding of formulas inside the story is, however, lifted by putting all formulas into the last chapter, so that the story can be read through. This is done for the sake of those who would not or could not follow university level math.

Still, as a novelty, the pdf file contains hypertext jumps to and back from the corresponding formulas. In the printed version, unfortunately, only the page number could have been indicated where to continue (or not) the reading.

The story is about (economical, wealth and social) inequality, and about the utopistic and distopic dream of treating equally persons with varying social and economical status. The mathematical models and notions behind are introductory to the area of statistics, statistical pyhsics, econophysics and quantitaive finance.

This book, as a novel mixture of a fiction and a theoretical textbook, may prove to be a chimera, but perhaps also a new dynamit for the intellectual considerations about how to measure, model and manipulate inequality in our societies. As said, an experiment.

Contact information

HUN-REN Wigner Research Centre for Physics
1121 Budapest, Konkoly-Thege Miklós út 29-33.

E-mail: wigner@wigner.hun.ren.hu
Website: wigner.hun-ren.hu

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